# A Modal Logic of Grounded Truth

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# Section 1

# Grounded Truth and the Ghost Challenge

Subsection 1

## Paradox and Groundedness

# Paradox

- Let $\Sigma$ be any $\mathcal{L}$-theory that interprets $\mathcal{L}$-syntax.

$$\text{(T)}\quad T\ulcorner\phi\urcorner \leftrightarrow \phi, \text{ for } \phi \in \mathcal{L}$$

- On pain of contradiction, we can't add every instance of (T) to $\Sigma$.
- We may ban '$T$' from $\mathcal{L}$ and ascend to a meta-language.
- Not so, however, for our universal theory.

# Groundedness, extensionally

- Let's restrict (T) to its grounded instances.
- What is groundedness?
- Kripke gave us an extensional characterization:
  - Let's focus on arithmetic, and its standard model $\mathfrak{N}$.
  - Let $\Gamma_m$ be an operator on sets of sentence such that

$$\phi \in \Gamma_m(X) \Leftrightarrow \mathfrak{N}(X) \vDash_m \phi$$

    e.g. $m =$SK, Strong Kleene
  - $\phi$ is grounded iff $\phi \in I_{\Gamma_{\mathrm{SK}}}$ (short: '$I_{\mathrm{SK}}$')
- Why is the theory of $\mathfrak{N}(I_{\mathrm{SK}})$ a good theory of truth?

# Groundedness, intuitively

- Initially, Alice speaks English minus ‘true’.
- Having learnt $\phi$, she infers that $\phi$ is true.
- And so on …
- $I^{+}_{\mathrm{SK}}$ models what Alice learns at some point.

# Groundedness



- Stripping off metaphor we get two core principles:
  1. $T\ulcorner\phi\urcorner$ presupposes $\phi$.
  2. $\phi$ *grounded* if its presuppositions bottom out in non-semantic sentences.
- $\mathfrak{N}(I_{\mathrm{SK}})$ captures this idea.
  1. $T\ulcorner\phi\urcorner$ true in $\mathfrak{N}(I_{\mathrm{SK}})$ only if $T\ulcorner\phi\urcorner$ true at some stage $\alpha + 1$ of the construction, only if $\phi$ true at stage $\alpha$.
  2. At stage 0, no sentence containing '$T$' is true.

# Subsection 2

## The Ghost of the Hierarchy

# A Challenge

- Although the truth predicate of Kripke's theory is type-free, the concept of groundedness is meta-theoretic.
- Hence, we cannot carry out the desired restriction of Tarski's schema to grounded truths in our own theory.

*[. . . ] the ghost of the Tarski hierarchy is still with us. (Kripke 1975:714)*

- The argument requires:
- *We cannot express groundedness by other means.*
- I will argue that we can.

# Ghost Challenge vs. Revenge

- The challenge I will address is distinct from what has been discussed as revenge.
  - "Using our object-language truth predicate, we cannot state the fact that the liar sentence is not (determinately) true."
- Revenge is about how much we can do with grounded truth.
- The ghost challenge is about whether we can use groundedness in the first place.

# Subsection 3

## Sidestepping the Ghost

# Expressing Groundedness 1

- My goal: formalizing the idea of groundedness without ascending to a meta-language.
- I formulated it in (philosophers') English:
  - $T\ulcorner\phi\urcorner$ presupposes $\phi$.
  - $\phi$ *grounded* if its presuppositions bottom out in non-semantic $\psi$.
- Maybe, 'presupposes' covers an implicit appeal to meta-theoretic resources.

# Expressing Groundedness 2

- But here's a way of putting it (schematically) in plain English:

  *For it to be true that $\phi$, it must have been the case that $\phi$ earlier.*

- We use tense to express the priority of $\phi$ over $T\ulcorner\phi\urcorner$.
- Similarly, we can express that presuppositions bottom out:

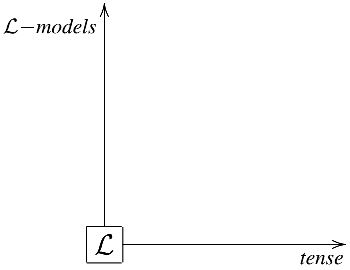
  *Once, nothing was true.*

# My Response

- We can express groundedness using tense.
- English already has tense.
- There is a non-meta-theoretic way of expressing groundedness.
- The friend of grounded truth is not forced up a hierarchy of theories.

# Sidestepping the Ghost

- What if our theory is formulated in a tense-free language?
- Let's add tense.
- This is not going meta-theoretic.



Section 2

# A Modal Logic of Grounded Truth

Subsection 1

# Tense Logic

# Adding Tense to Truth over Arithmetic

- Let $\mathcal{L}_{\mathrm{at}}$ be the language of first order arithmetic extended by a unary relation symbol '$T$'.
- I add the resources of tense logic.
- Two primitive operators:
  - $\boldsymbol{H}\phi$: *it was always the case that* $\phi$
  - $\boldsymbol{G}\phi$: *it will always be the case that* $\phi$
- Defined symbols
  - $\boldsymbol{P}\phi :\Leftrightarrow \neg\boldsymbol{H}\neg\phi$ : *it was the case that* $\phi$
  - $\boldsymbol{F}\phi :\Leftrightarrow \neg\boldsymbol{G}\neg\phi$: *it will be the case that* $\phi$

# The Logic of Well-Ordered Time 1

- Necessitation for $\boldsymbol{G}$ and $\boldsymbol{H}$.

K $\boldsymbol{G}$ and $\boldsymbol{H}$ distribute over conditionals.

- $\phi \to \boldsymbol{GP}\phi$
- $\phi \to \boldsymbol{HF}\phi$

$4_{\boldsymbol{G}}$ $\boldsymbol{G}\phi \to \boldsymbol{GG}\phi$

$.3_{\boldsymbol{P}}$ $\boldsymbol{P}\phi \wedge \boldsymbol{P}\psi \to \boldsymbol{P}(\phi \wedge \boldsymbol{P}\psi) \vee \boldsymbol{P}(\phi \wedge \psi) \vee \boldsymbol{P}(\boldsymbol{P}\phi \wedge \psi)$

$.3_{\boldsymbol{F}}$ $\boldsymbol{F}\phi \wedge \boldsymbol{F}\psi \to \boldsymbol{F}(\phi \wedge \boldsymbol{F}\psi) \vee \boldsymbol{F}(\phi \wedge \psi) \vee \boldsymbol{F}(\boldsymbol{F}\phi \wedge \psi)$

$\mathrm{L}_{\boldsymbol{H}}$ $\boldsymbol{H}(\boldsymbol{H}\phi \to \phi) \to \boldsymbol{H}\phi$

# The Logic of Well-Ordered Time 2





- Only truth changes "over time": domain and interpretation of terms is constant.
- Classical, non-free quantification.

RT $\dfrac{s=t}{\boldsymbol{G}s=t\wedge\boldsymbol{H}s=t}$ $\dfrac{s\neq t}{\boldsymbol{G}s\neq t\wedge\boldsymbol{H}s\neq t}$

- We get a simple quantified logic of well-ordered time: "*woq*".

# Subsection 2

# Tensed Truth

# Base Theory

- I now give axioms for a tensed theory of truth.
- Let's define:
    - $\boldsymbol{S}\phi$: $\boldsymbol{P}\phi \vee \phi \vee \boldsymbol{F}\phi$ *sometimes*
    - $\boldsymbol{A}\phi$: $\boldsymbol{H}\phi \wedge \phi \wedge \boldsymbol{G}\phi$ *always*
- Base theory PA, marked as being always the case.

# The Ground

- $S\neg\exists xTx$: *Once, nothing was true.*

# Truth Introduction 1

- How do sentences become true?
- My goal is groundedness as given by Kripke's Strong Kleene ('SK') construction.
- *Needed*: Axioms stating that the extension of '$T$' grows according to the SK jump.

# Truth Introduction 2

- *Problem*: Our base logic of well-ordered time is classical.
- I need axioms that express in classical logic truth introduction according to the SK jump.
- The *Kripke-Feferman* axioms ('KF') describe an SK fixed point.
- *Solution*: Dynamize KF.

# Tensing KF 1

- (TKF1)
  $\boldsymbol{A}\forall x \forall y((Tx \dot{=} y \rightarrow \boldsymbol{P}x = y) \wedge (x = y \rightarrow \boldsymbol{F}\, Tx \dot{=} y \wedge \boldsymbol{G}\, Tx \dot{=} y))$

# Tensing KF 1

- (TKF1) $\boldsymbol{A}\forall x\forall y((Tx\dot{=}y \rightarrow \boldsymbol{P}x = y) \wedge (x = y \rightarrow \boldsymbol{F}\,Tx\dot{=}y \wedge \boldsymbol{G}\,Tx\dot{=}y))$
- (TKF2) $\boldsymbol{A}\forall x\forall y((Tx\dot{\neq}y \rightarrow \boldsymbol{P}x \neq y) \wedge (x \neq y \rightarrow \boldsymbol{F}\,Tx\dot{\neq}y \wedge \boldsymbol{G}\,Tx\dot{\neq}y))$

# Tensing KF 1

- (TKF1)
  $\boldsymbol{A}\forall x \forall y\big((Tx \underset{\cdot}{=} y \rightarrow \boldsymbol{P} x = y) \wedge (x = y \rightarrow \boldsymbol{F}\, Tx \underset{\cdot}{=} y \wedge \boldsymbol{G}\, Tx \underset{\cdot}{=} y)\big)$
- (TKF2)
  $\boldsymbol{A}\forall x \forall y\big((Tx \underset{\cdot}{\neq} y \rightarrow \boldsymbol{P} x \neq y) \wedge (x \neq y \rightarrow \boldsymbol{F}\, Tx \underset{\cdot}{\neq} y \wedge \boldsymbol{G}\, Tx \underset{\cdot}{\neq} y)\big)$
- (TKF12) $\boldsymbol{A}\forall x\big((T\underset{\cdot}{T}x \rightarrow \boldsymbol{P}Tx) \wedge (Tx \rightarrow \boldsymbol{F}\, T\underset{\cdot}{T}x \wedge \boldsymbol{G}\, T\underset{\cdot}{T}x)\big)$

# Tensing KF 1

- (TKF1) $\boldsymbol{A}\forall x\forall y\big((Tx\underset{\cdot}{=}y \rightarrow \boldsymbol{P}x = y) \wedge (x = y \rightarrow \boldsymbol{F}\,Tx\underset{\cdot}{=}y \wedge \boldsymbol{G}\,Tx\underset{\cdot}{=}y)\big)$
- (TKF2) $\boldsymbol{A}\forall x\forall y\big((Tx\underset{\cdot}{\neq}y \rightarrow \boldsymbol{P}x \neq y) \wedge (x \neq y \rightarrow \boldsymbol{F}\,Tx\underset{\cdot}{\neq}y \wedge \boldsymbol{G}\,Tx\underset{\cdot}{\neq}y)\big)$
- (TKF12) $\boldsymbol{A}\forall x\big((T\underset{\cdot}{T}x \rightarrow \boldsymbol{P}Tx) \wedge (Tx \rightarrow \boldsymbol{F}\,T\underset{\cdot}{T}x \wedge \boldsymbol{G}\,T\underset{\cdot}{T}x)\big)$
- (TKF13) $\boldsymbol{A}\forall x\big((T\underset{\cdot}{\neg}\underset{\cdot}{T}x \rightarrow (\boldsymbol{P}T\underset{\cdot}{\neg}x \vee \neg \underset{\cdot}{Sent}_{at}x)) \wedge ((T\underset{\cdot}{\neg}x \vee \neg \underset{\cdot}{Sent}_{at}x) \rightarrow \boldsymbol{F}\,T\underset{\cdot}{\neg}\underset{\cdot}{T}x \wedge \boldsymbol{G}\,T\underset{\cdot}{\neg}\underset{\cdot}{T}x)\big)$

# Tensing KF 2

- Finally, we add those KF axioms that govern how ‘$T$’ interacts with $\wedge$, $\vee$, $\exists$ and $\forall$.
- Truth is closed under Strong Kleene logic at every stage.
- Therefore, we take KF3-KF11 and put an ‘$A$’ in front.
- For example:

$$\text{TKF5} \quad \mathbf{A}\forall x \forall y \big(Se\underset{\cdot}{n}t_{\text{at}}(x \underset{\cdot}{\wedge} y) \rightarrow \big(T \underset{\cdot}{\neg}(x \underset{\cdot}{\wedge} y) \leftrightarrow T \underset{\cdot}{\neg} x \vee T \underset{\cdot}{\neg} y\big)\big)$$

# A Modal Logic of Grounded Truth

MGT:= Always PA + Once $\neg\exists x\, Tx$ + dynamized KF ("truth increases over time according to the Strong Kleene jump")

# Nothing is Lost...

- How does MGT relate to standard, non-modal KF?
- Let $(Tx)^\star = STx$
- Translate arithmetic, connectives and quantifiers homophonically.

## Proposition

MGT interprets KF.

$$\mathrm{KF} \vdash \phi \Rightarrow \mathrm{MGT} \vdash_{woq} (\phi)^\star$$

# . . . and Much is Gained

## Proposition

The modal logic of grounded truth proves the necessary consistency of truth.

$$\mathrm{MGT} \vdash_{woq} A\forall x\big(\underset{\cdot}{Sent}_{\mathrm{at}}(x) \rightarrow \neg(Tx \wedge \neg T\underset{\cdot}{\neg}x)\big)$$

(*Proof idea*) Induction on well-ordered tense: at least point, nothing is true. At induction step, assume otherwise, reason from $T\ulcorner\phi\urcorner \wedge T\ulcorner\neg\phi\urcorner$ to that at some earlier stage $\phi \wedge \neg\phi$, contradiction.

# The Truth-Teller is Never True

## Proposition

Let $\tau$ be a *truth-teller*, such that PA$\vdash \tau \leftrightarrow T\ulcorner\tau\urcorner$. Then

$$\mathrm{MGT} \vdash_{woq} \neg ST\ulcorner\tau\urcorner$$

(*Proof idea*) Thanks to tensed truth, we can formalize the intuitive reasoning: Assume that $T\ulcorner\tau\urcorner$ at some point, then there's an earliest such point, at which it must have been the case that $\tau$ earlier. Contradiction.

Subsection 3

# Tensed Truth and the Stages of Kripke's Construction

# Standard Numbers

- Of course, first-order PA is incomplete: MGT will have non-standard models.
- But this is orthogonal to whether MGT captures groundedness.
- We're entitled to help ourselves to standard arithmetic.
- Let's identify the "worlds" with models $\mathfrak{N}(X)$.

# MGT Worlds are Kripke Stages

## Definition (KC)

Let KC be the set of models $\mathfrak{N}(I_{\mathrm{SK}}^{+,\alpha})$, $\alpha < \omega_1^{\mathrm{CK}}$, well-ordered by the relation of proper subsethood $\subset$ on the extensions $I_{\mathrm{SK}}^{+,\alpha}$.

## Proposition (Adequacy)

For every *woq*-frame $(W, \prec)$ such that $W$ is a set of models $\mathfrak{N}(X)$,

$$\forall w \in W \ (W, \prec) \models \mathrm{MGT}[w] \text{ if and only if } (W, \prec) = \mathrm{KC}$$

# Expressing Groundedness

## Definition

Let us write "$\Sigma \models_{\mathfrak{N}} \phi$" iff for every set $W$ of models $\mathfrak{N}(X)$ well-ordered by $\prec$, and for every model $w \in W$,

$$(W, \prec) \models \Sigma[w] \Rightarrow (W, \prec) \models \phi[w]$$

- Recall that $I^{+}_{\mathrm{SK}}$ is the extension of the Strong Kleene fixed point – the set of grounded truths.

## Corollary

For every $\mathcal{L}_{\mathrm{at}}$-sentence $\phi$,

$$\ulcorner\phi\urcorner \in I^{+}_{\mathrm{SK}} \Leftrightarrow \mathrm{MGT} \models_{\mathfrak{N}} \boldsymbol{S}\, T\ulcorner\phi\urcorner$$

# Section 3

## Conclusion

# Conclusion



- The groundedness approach to truth faces a challenge: "groundedness is a meta-theoretic notion".
- I proposed a response: Express groundedness using tense.

1. For it to be true that $\phi$, it must have been the case that $\phi$ earlier.
2. Once, nothing was true.

- I presented one implementation of this proposal:
- MGT $:=$ Always PA + Once $\neg\exists x\, Tx$ + dynamized KF
- MGT characterizes the stages of Kripke's construction.

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