A Modal Logic of Grounded Truth

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Section 1

Grounded Truth and the Ghost Challenge

Subsection 1

Paradox and Groundedness

Paradox

- Let Σ be any $\mathcal{L}$-theory that interprets syntax.

  (T) $T\ulcorner\phi\urcorner \leftrightarrow \phi$

- On pain of contradiction, we can't add every instance of (T) to Σ.
- We may ban 'T' from $\mathcal{L}$ and ascend to a meta-language.
- Not so, however, for our universal theory.

Groundedness 1

- Let's restrict (T) to its grounded instances.
- What is groundedness?
- Kripke gave us an extensional characterization:
  - Let's focus on arithmetic, and its standard model $\mathfrak{N}$.
  - Let Γ_m be an operator on sets of sentence such that

$$\phi \in \Gamma_m(X) \Leftrightarrow \mathfrak{N}(X) \vDash_m \phi$$

    e.g. $m =$SK, Strong Kleene
  - ϕ is grounded iff $\phi \in I_{\Gamma_m}$ (short: 'I_m')
- Why is the theory of $\mathfrak{N}(I_{\mathrm{SK}})$ a good theory of truth?

Groundedness 2

- Informal gloss:
  - $T\ulcorner\phi\urcorner$ presupposes ϕ.
  - ϕ *grounded* if its presuppositions bottom out in non-semantic sentences.
- $\mathfrak{N}(I_{\mathrm{SK}})$ captures this intuitive idea.
  - $T\ulcorner\phi\urcorner$ true in $\mathfrak{N}(I_{\mathrm{SK}})$ only if $T\ulcorner\phi\urcorner$ true at some stage $\alpha + 1$ of the construction, only if ϕ true at stage α.
  - At stage 0, no sentence containing 'T' is true.

Subsection 2

The Ghost of the Hierarchy

A Challenge

- Although the truth predicate of Kripke's theory is type-free, the concept of groundedness is meta-theoretic.
- Hence, we cannot carry out the desired restriction of Tarski's schema to grounded truths in our own theory.

[. . .] the ghost of the Tarski hierarchy is still with us. (Kripke 1975:714)

- The argument requires:
- *We cannot express groundedness by other means*.
- I will argue that we can.

Ghost Challenge vs Revenge

- The challenge I will address is distinct from what has been discussed as revenge.
  - "Using our object-language truth predicate, we cannot state the fact that the liar sentence is not (determinately) true."
- Revenge is about how much we can do with grounded truth.
- The ghost challenge is about whether we can use groundedness in the first place.

Subsection 3

Sidestepping the Ghost

Expressing the Informal Idea 1

- My goal: formalizing the idea of groundedness without ascending to a meta-language.
- I formulated it in (philosophers') English:
  - $T\ulcorner\psi\urcorner$ presupposes ψ.
  - ϕ *grounded* if its presuppositions bottom out in non-semantic ψ.
- Maybe, 'presupposes' covers an implicit appeal to meta-theoretic resources.

Expressing the Informal Idea 2

- But here's a way of putting it (schematically) in plain English:

  For it to be true that ϕ, it must have been the case that ϕ earlier.

- We use tense to express the priority of $T\ulcorner\phi\urcorner$ over ϕ.
- Similarly, we can express that presuppositions bottom out.

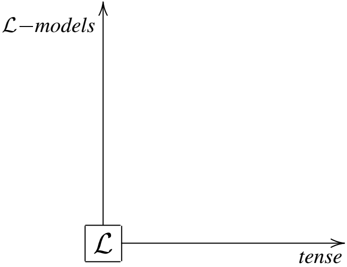
  Once, nothing was true.

My Response

- We can express groundedness using tense.
- English already has tense.
- There is a non-meta-theoretic way of expressing groundedness.
- The friend of grounded truth is not forced up a hierarchy of theories.

Sidestepping the Ghost

- What if our theory is formulated in a tense-free language?
- Let's add tense.
- This is not going meta-theoretic.



Section 2

A Modal Logic of Grounded Truth

Subsection 1

Tense Logic

Adding Tense to Truth over Arithmetic

- Let $\mathcal{L}_{\text{at}}$ be the language of first order arithmetic extended by a unary relation symbol 'T'.
- I add the resources of tense logic.
- Two primitive operators:
  - $\boldsymbol{H}\phi$: *it has always been the case that* ϕ
  - $\boldsymbol{G}\phi$: *it will always be the case that* ϕ
- Defined symbols
  - $\boldsymbol{P}\phi :\Leftrightarrow \neg\boldsymbol{H}\neg\phi$: *it was the case that* ϕ
  - $\boldsymbol{F}\phi :\Leftrightarrow \neg\boldsymbol{G}\neg\phi$: *it will be the case that* ϕ

The Logic of Well-Ordered Time 1

- Necessitation for $\boldsymbol{G}$ and $\boldsymbol{H}$.

K $\boldsymbol{G}$ and $\boldsymbol{H}$ distribute over conditionals.

- $\phi \to \boldsymbol{GP}\phi$
- $\phi \to \boldsymbol{HF}\phi$

4_G $\boldsymbol{G}\phi \to \boldsymbol{GG}\phi$

$.3_P$ $\boldsymbol{P}\phi \wedge \boldsymbol{P}\psi \to \boldsymbol{P}(\phi \wedge \boldsymbol{P}\psi) \vee \boldsymbol{P}(\phi \wedge \psi) \vee \boldsymbol{P}(\boldsymbol{P}\phi \wedge \psi)$

$.3_F$ $\boldsymbol{F}\phi \wedge \boldsymbol{F}\psi \to \boldsymbol{F}(\phi \wedge \boldsymbol{F}\psi) \vee \boldsymbol{F}(\phi \wedge \psi) \vee \boldsymbol{F}(\boldsymbol{F}\phi \wedge \psi)$

L_H $\boldsymbol{H}(\boldsymbol{H}\phi \to \phi) \to \boldsymbol{H}\phi$

- Following Burgess (1984, §2.8), let's call this logic L_8.

The Logic of Well-Ordered Time 2

- Only truth changes "over time": domain and interpretation of terms is constant.

BF$_G$ $\forall x \boldsymbol{G}\phi \rightarrow \boldsymbol{G}\forall x\phi$

BF$_H$ $\forall x \boldsymbol{H}\phi \rightarrow \boldsymbol{H}\forall x\phi$

RT $\frac{s=t}{\boldsymbol{G}s=t \wedge \boldsymbol{H}s=t}$

- Following Garson (1984, §2.8) I refer to the result as Q1L_8.

Subsection 2

Tensed Truth

Base Theory

- I now give axioms for a tensed theory of truth.
- Let's define:
  - $\boldsymbol{S}\phi$: $\boldsymbol{P}\phi \vee \phi \vee \boldsymbol{F}\phi$ *sometimes*
  - $\boldsymbol{A}\phi$: $\boldsymbol{H}\phi \wedge \phi \wedge \boldsymbol{G}\phi$ *always*
- Base theory PA, marked as being always the case.

The Ground

- $S\neg\exists xTx$: *Once, nothing was true*.

Truth Axioms

- How do sentences become true?
- My goal is groundedness as given by Kripke's Strong Kleene ('SK') construction.
- Needed: Axioms stating that the extension of 'T' grows according to the SK jump.
- Problem: Our base logic of well-ordered time is classical.
- I need axioms that express in classical logic truth introduction according to the SK jump.
- The *Kripke-Feferman* axioms ('KF') describe an SK fixed point.
- Solution: Dynamize KF.

Tensing KF 1

- (GKF1)
$\boldsymbol{A}\forall x \forall y((Tx \dot{=} y \rightarrow \boldsymbol{P} x = y)) \wedge (x = y \rightarrow \boldsymbol{F}\, Tx \dot{=} y \wedge \boldsymbol{G}\, Tx \dot{=} y)$

Tensing KF 1

- (GKF1)
  $\boldsymbol{A}\forall x \forall y((Tx \underset{\cdot}{=} y \to \boldsymbol{P} x = y)) \wedge (x = y \to \boldsymbol{F}\, Tx \underset{\cdot}{=} y \wedge \boldsymbol{G}\, Tx \underset{\cdot}{=} y)$
- (GKF2)
  $\boldsymbol{A}\forall x \forall y((Tx \underset{\cdot}{\neq} y \to \boldsymbol{P} x \neq y) \wedge (x \neq y \to \boldsymbol{F}\, Tx \underset{\cdot}{\neq} y \wedge \boldsymbol{G}\, Tx \underset{\cdot}{\neq} y))$

Tensing KF 1

- (GKF1) $\boldsymbol{A}\forall x\forall y((Tx\underset{\cdot}{=}y \rightarrow \boldsymbol{P}x = y)) \wedge (x = y \rightarrow \boldsymbol{F}\,Tx\underset{\cdot}{=}y \wedge \boldsymbol{G}\,Tx\underset{\cdot}{=}y)$
- (GKF2) $\boldsymbol{A}\forall x\forall y((Tx\underset{\cdot}{\neq}y \rightarrow \boldsymbol{P}x \neq y) \wedge (x \neq y \rightarrow \boldsymbol{F}\,Tx\underset{\cdot}{\neq}y \wedge \boldsymbol{G}\,Tx\underset{\cdot}{\neq}y))$
- (GKF12) $\boldsymbol{A}\forall x((T\underset{\cdot}{T}x \rightarrow \boldsymbol{P}Tx) \wedge (Tx \rightarrow \boldsymbol{F}\,T\underset{\cdot}{T}x \wedge \boldsymbol{G}\,T\underset{\cdot}{T}x))$

Tensing KF 1

- (GKF1)
  $\boldsymbol{A}\forall x\forall y((Tx\underset{\cdot}{=}y \rightarrow \boldsymbol{P}x = y)) \wedge (x = y \rightarrow \boldsymbol{F}\,Tx\underset{\cdot}{=}y \wedge \boldsymbol{G}\,Tx\underset{\cdot}{=}y)$
- (GKF2)
  $\boldsymbol{A}\forall x\forall y((Tx\underset{\cdot}{\neq}y \rightarrow \boldsymbol{P}x \neq y) \wedge (x \neq y \rightarrow \boldsymbol{F}\,Tx\underset{\cdot}{\neq}y \wedge \boldsymbol{G}\,Tx\underset{\cdot}{\neq}y))$
- (GKF12) $\boldsymbol{A}\forall x((T\underset{\cdot}{T}x \rightarrow \boldsymbol{P}Tx) \wedge (Tx \rightarrow \boldsymbol{F}\,T\underset{\cdot}{T}x \wedge \boldsymbol{G}\,T\underset{\cdot}{T}x))$
- (GKF13) $\boldsymbol{A}\forall x((T\underset{\cdot}{\neg}\underset{\cdot}{T}x \rightarrow (\boldsymbol{P}T\underset{\cdot}{\neg}x \vee \underset{\cdot}{Sent}_{\text{at}}(x))) \wedge ((T\underset{\cdot}{\neg}x \vee \underset{\cdot}{Sent}_{\text{at}}(x)) \rightarrow \boldsymbol{F}\,T\underset{\cdot}{\neg}\underset{\cdot}{T}x \wedge \boldsymbol{G}\,T\underset{\cdot}{\neg}\underset{\cdot}{T}x))$

Tensing KF 2

- Finally, we add those KF axioms that govern how 'T' interacts with $\wedge$, $\vee$, $\exists$ and $\forall$.
- Truth is closed under Strong Kleene logic at every stage.
- Therefore, we take KF3-KF11 and put an $\boldsymbol{A}$ in front.
- For example:

GKF5 $\boldsymbol{A}\forall x \forall y\,(Sent_{at}(x \dot{\wedge} y) \rightarrow (T \dot{\neg}\,(x \dot{\wedge} y) \leftrightarrow T \dot{\neg} x \vee T \dot{\neg} y))$

A Modal Logic of Grounded Truth

MGT:= always PA + At some time $\neg\exists x(Tx)$ + GKF ("truth increases over time according to the Strong Kleene jump")

Nothing is Lost. . .

- How does MGT relate to standard, non-modal KF?
- Let $(Tx)^{\star} = \boldsymbol{S}Tx$, translate arithmetic, connectives and quantifiers homophonically.

Proposition

MGT interprets KF.

$$\mathrm{KF} \vdash \phi \Rightarrow \mathrm{MGT} \vdash_{Q1L8} (\phi)^{\star}$$

… and Much is Gained

- But MGT is stronger than KF:

Proposition

The modal logic of grounded truth proves the necessary consistency of truth.

$$\text{MGT} \vdash_{Q1L8} A\forall x(\underset{.}{Sent}_{\text{at}}(x) \to \neg(Tx \wedge \neg T \underset{.}{\neg} x))$$

(*Proof idea*) Induction on well-ordered tense: at least point, nothing is true. At induction step, assume otherwise, reason from $T\ulcorner\phi\urcorner \wedge T\ulcorner\neg\phi\urcorner$ to that at some earlier stage $\phi \wedge \neg\phi$, contradiction.

The Truth-Teller is Never True

Proposition

Let τ be a *truth-teller*, such that PA$\vdash \tau \leftrightarrow T\ulcorner\tau\urcorner$. Then

$$\mathrm{MGT} \vdash_{Q1L8} \neg ST\ulcorner\tau\urcorner$$

(*Proof idea*) Thanks to tensed truth, we can formalize the intuitive reasoning: Assume that $T\ulcorner\tau\urcorner$ at some point, then there's an earliest such point, at which it must have been the case that τ earlier. Contradiction.

Burgess's KFB

- So much can be done by adding a minimality scheme to KF, as Burgess showed recently.
  - KFB:= KF+"If ϕ satisfies the KF-axioms, then $\forall x(\phi(x) \rightarrow Tx)$".

Question

How does MGT relate to KFB?

MGT vs KFB

Fact (Halbach)

KFB holds in other models than the least fixed point.

- MGT does better, precisely because it is a tensed theory of truth.

Subsection 3

Tensed Truth and the Stages of Kripke's Construction

Standard Numbers

- Of course, first-order PA is incomplete: MGT will have non-standard models.
- But this is orthogonal to whether MGT captures groundedness.
- We're entitled to help ourselves to standard arithmetic.
- Let's identify the "worlds" with models $\mathfrak{N}(X)$.

MGT Worlds are Stages

Definition (KC)

Let KC be the set of models $\mathfrak{N}(I_{\text{SK}}^{+,\alpha})$, $\alpha < \omega_1^{\text{CK}}$, well-ordered by the relation of proper subsethood $\subset$ on the extensions $I_{\text{SK}}^{+,\alpha}$.

Proposition (Adequacy)

For every Q1L$_8$-frame $(W, \prec)$ such that W is a set of models $\mathfrak{N}(X)$,

$$\forall w \in W\ (W, \prec) \models \text{MGT}[w] \text{ if and only if } (W, \prec) = \text{KC}$$

Expressing Groundedness

Definition

Let us write "$\Sigma \models_{\mathfrak{N}} \phi$" iff for every set W of models $\mathfrak{N}(X)$ well-ordered by $\prec$, and for every model $w \in W$,

$$(W, \prec) \models \Sigma[w] \Rightarrow (W, \prec) \models \phi[w]$$

- Recall that I^{+}_{SK} is the extension of the Strong Kleene fixed point – the set of grounded truths.

Corollary

For every $\mathcal{L}_{\text{at}}$-sentence ϕ,

$$\ulcorner\phi\urcorner \in I^{+}_{\text{SK}} \Leftrightarrow \text{MGT} \models_{\mathfrak{N}} \boldsymbol{S}\,T\ulcorner\phi\urcorner$$

Section 3

Conclusion

Conclusion

- The groundedness approach to truth faces a challenge: "groundedness is a meta-theoretic notion".
- I proposed a response: Express groundedness using tense.

1. For ϕ to be true, it must have been the case that ϕ earlier.
2. At some point, nothing was true.

- I presented one implementation of this proposal:
- Tensed KF characterizes the stages of Kripke's construction.

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