How do we respond to paradox? A major contender is the idea of *grounded* truth. This talk is about a problem for this approach, and how to solve it by going modal.

My talk has two parts, one philosophical, the other somewhat more technical. I start out by motivating the groundedness approach to type-free truth. Then, I will present the following challenge. Groundedness itself is defined meta-theoretically, such that "[...] the ghost of the hierarchy is still with us". I argue that this challenge can be answered. I propose to express groundedness by intensional means.

Finally, I show how to implement this proposal for truth over first order arithmetic.

...

0.0.1 Paradox

This won't work unless we somehow restrict which sentences are inserted for φ. For example, we may ban the truth predicate altogether and ascend to a meta-language with a predicate of object-language-truth. There, (T) can be stated safely for truth-free sentences. But as a theory of truth, this restricted T-schema is rather frustrating. Here's one problem.

We would like to speak about truth in English. But English does not have a meta-language: Everything that can be at all spoken of, we can speak of in English.

This may be challenged. At any rate, we would like a universal, all-encompassing theory. From it, by definition, we cannot ascend to a meta-theory.

Isn't there a better way of responding to paradox, a more sophisticated restriction of (T) that allows us to speak of truth in our own language?

0.0.2 Groundedness 1

Yes, there is. We can restrict (T) to the grounded sentences. What do I mean by "grounded"? Part of what I want to do today is to work on a better understanding of groundedness. But we already have some grip on the extension of the notion.

Kripke showed us how to expand a given base model by a set of sentences *inductive* in this base model, using an operator that turns truth in a model into a new model.

0.0.3 Groundedness 2

Why is this a good model of truth? Well, for one, we get Tarski's schema for every sentence in the least fixed point. But why is *this* the right restriction?

I propose the following answer. Here's a pre-theoretic gloss on groundedness. First, the truth of ψ presupposes ψ. Second, this presupposition must bottom out in non-semantic truths.

The least fixed point model captures this intuitive idea. First, $\mathrm{T}\varphi$ holds in the model only if it holds at some stage of the construction only if φ holds at an earlier stage. Second, there is a purely non-semantic bottom stage.

0.1 The Ghost of the Hierarchy

This groundedness approach to truth faces a challenge.

0.1.1 A Challenge

It goes like this. The notion of groundedness that does the work for us is itself essentially meta-theoretic. Therefore, we cannot state the proposed restriction of Tarski's schema in our own language.

This challenge has some force. The usual characterizations of groundedness really are meta-theoretic. Kripke himself famously noted this:

> The ghost of the Tarski hierarchy is still with us.

But note that the argument requires that there is no way of expressing groundedness other than the usual, meta-theoretic way. This is a strong assumption. My goal in the following is to give reasons to doubt it.

But before I do that, let me say that this challenge, which I want to answer, is distinct

0.1.2 Not: Revenge

from the problem of *revenge*, which I won't attempt to solve. By *revenge*, I mean the objection that using our predicate of grounded truth, we cannot say that the liar sentence is not true.

To see that this is a separate problem, assume, just for the moment, that we accept revenge. That is, we accept that if we look at our theory from the outside, there are facts pertaining to truth, which cannot express using the truth predicate of our theory.

Now, even if we accept this, the challenge from groundedness being a meta-theoretic notion is still pressing. Since, if we're in our universal theory, we won't have a chance even to carry out the desired restriction of Tarski's schema.

Revenge is about how much we can do with grounded truth – whereas the challenge that I want to address is whether we can use groundedness in the first place.

0.2 Sidestepping the Ghost

So my goal is to develop a way of expressing groundedness that is not meta-theoretic.

0.2.1 Expressing the Informal Idea 1

And really, this should be possible. After all, I've communicated the intuitive idea to you in our own language, just now. The truth of φ presupposes φ. You may object that "presupposition", if spelt out properly, forces you to a meta-theory.

0.2.2 Expressing the Informal Idea 2

But I didn't have to use this term. I could as well, and probably more naturally, have said:

> For φ to be true, it must have been the case that φ earlier.

What about the second component of groundedness, that presupposition must bottom out. This is somewhat harder to come by, but here's one way of doing it:

> Once, nothing was true.

0.2.3 My Response

So here's my response to the challenge. There is a way of expressing groundedness that is not meta-theoretic. Instead, we can use *tense*. And English already has tense, and so would, arguably, a universal language.

0.2.4 Sidestepping the Ghost

But even if this wasn't the case, and we'd have to add temporal operators, this would still not mean to go meta-theoretic.

Let me use the following picture. Adding the meta-theoretic resources is a *vertical* extension of our language: My proposal is that of a *horizontal* enrichment. We don't have to let the ghost chase us up the hierarchy: Let's sidestep it.

1 A Modal Logic of Grounded Truth

I now turn to implement my proposal.

1.1 Tense Logic

I'll formulate a theory of truth based on a basic logic of time. But let me note that is really just one way of carrying out my idea, and I don't think my philosophical proposal stands or falls by it.

1.1.1 Adding Tense to Truth over Arithmetic

So consider the language of arithmetic extended by a unary relation symbol 'T'

What axioms govern these primitives? The second component of groundedness is that presupposition is well-ordered. I want to express presupposition using tense. Consequently, I choose an axiom system that characterizes well-ordered time.

1.1.2 The Logic of Well-Ordered Time

The basic logic of time, basically K for both G and H, is extended by the axiom 4, to make time a partial order, by axioms .3 both for past and future, to rule out branching and ensure a linear ordering, and finally Löb's axiom for strong past, which makes time well-founded.

For my audience today I've looked up how this system is called in Burgess' handbook entry, it's called L-eight.

1.1.3 The Logic of Well-Ordered Time 1

Quantified modal logic is hard, both technically and philosophically. Fortunately, I don't really have to deal with it. All I want to say is that *truth* changes over time. The domain and how terms are interpreted I want to keep constant.

1.2 Tensed Truth

1.2.1 Base Theory

I now formulate a theory of truth in this logic of time. It is based on first-order arithmetic. Again, it's only truth that changes over time, so I prefix the PA axioms by a defined operator reading “always”.

1.2.2 The Ground

As we have the base theory, we add one axiom to the effect that at some point, all there is to say This corresponds to how I formulated the second component of groundedness, well-foundedness constraint on presupposition: once, nothing was true.

1.2.3 Truth Axioms

Then, more and more sentences became true. Now I need axioms to govern this introduction of truth. My goal is specific. I want to capture the notion of groundedness of Kripke's model contstruction, based on Strong Kleene logic. So I want truth to be introduced according to the Strong Kleene jump operator. But, my logic of tense is classical. So, I need axioms that express truth introduction according to the Strong Kleene jump operator, and do so in classical logic.

The KF axioms describe a Strong Kleene *fixed point*. What I'll do, therefore, is that I'll use my tense operators, to turn KF into axioms of step-by-step truth introduction.

1.2.4 Tensing KF 1

Here we go...

x equals y is true only if earlier, it's been the case that x=y but if x equals y then this will be true, and in fact from now on.

1.2.5 Tensing KF 2

What about the connectives and quantifiers? At every stage, truth is closed under Strong Kleene logic. Hence, we can add the remaining KF axioms as they are, merely putting an “always” in front.

1.2.6 MGT

This is my modal logic of grounded truth. Always arithmetic, the axiom of ground, and KF turned into axioms of step-by-step truth introduction.

1.2.7

If we translate truth as "being true at some point", we can interpret KF. So, nothing has gone lost by going tense.

1.2.8 Grounded Truth is Consistent

On the contrary, we have gained in strength. KF does not prove the consistency of truth. This is why often, consistency is added as an extra axiom.

In our present tense setting, however, we get consistency for free.

1.2.9 The Truth-Teller is Never True

Even more exciting: For every truth teller we can prove that it's never true. This is something that not even KF plus consistency can do.

1.2.10 Burgess' KFB

Here's a slide for Jimmy and Jack. Since, consistency and negated truth teller we also get without tense logic, as Burgess has shown.

1.2.11 MGT vs KFB

But, if we're after groundedness, Burgess' theory is not fully satisfactory. Since, it does not characterize the least fixed point model.

My tensed theory does somewhat better, as I'll lay out in the rest of my talk.

1.3 Tensed Truth and the Stages of Kripke's Construction

The main result is that my modal logic of grounded truth has a natural model: namely, the frame made up of the stages of Kripke's construction. It characterizes this frame exactly: it is categorical, in a relevant sense.

1.3.1 Standard Numbers

How can this be? After all, the theory is based on a merely first-order theory of arithmetic, whereas the Kripke construction builds on the *standard* model. And of course, first-order Peano Arithmetic can't single out the standard model.

As much as this is true, however, it is also irrelevant for the question whether the truth axioms characterize Kripke's construction.

So, I'll focus on models that interpret arithmetic the standard way. We can think of the worlds as expansions of the standard model.

MGT characterizes the stages just as well as KF characterizes the least fixed point.

1.3.2 MGT Worlds are Stages

Making this assumption, we can show categoricity in the following sense:

For every frame of the logic, and every interpretation that keeps arithmetic standard, exactly the structure of Kripke's stage models MGT.

1.3.3 Expressing Groundedness

As a corollary, we get that the theory expresses groundedness in the precise sense that phi is true at some point just in case it is grounded.

2 Conclusion

Let me wrap up.

The project of motivating a theory of type-free truth from the notion of groundedness faces the challenge that groundedness is a meta-theoretic notion. I offered a response to this challenge. We can express the idea of groundedness in our own language using intensional means, more precisely tense.

I presented one way of implementing this response and formulated a theory of truth based on the logic of well-ordered time. This axiomatic theory relates naturally to Kripke's semantic construction. I take this to be evidence for my proposal.