# Irregular but Grounded Class-Theory

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## 1 Introduction

Fine (2005) constructs models of a class theory closed under Boolean operations. He requires these models to be *regular*. In this paper, I argue that this requirement can be lifted. I propose and defend a generalization of Fine's construction: models of irregular but *grounded* classes.

The paper has the following structure. First, I introduce to Fine's class theory. I will explain the intuitive idea behind it (§2) as well as its formal implementation (3. Then, I explain the requirement of *regularity* that Fine imposes on the models (§4).

Fine takes regularity to ensure that the relation '... is defined in terms of ...' is well-founded on the classes. However, as I will argue in section 5, definitions of *irregular* models are no less well-founded.

In section 6, I will explore the philosophical dimension of these technical findings. I will argue that the class definitions of irregular models, too, are well-founded because their membership relation is *grounded* in ordinary set-theory.

## 2 Fine's Theory of Classes (Old)

Usually, when her goal is to extend ordinary set theory ZFC by classes, the theorist starts from the sets, and then builds up the classes, going from some objects to the class that they are the members of. In other words, she uses the notion of membership in order to obtain the realm of classes. Fine takes the opposite approach. He starts from the classes, and then, step by step, he defines the membership relation, on the basis of a fixed domain of classes.

The idea behind Fine's construction can be given by the following picture. Imagine a dialogue between God and archangel Gabriel. Assume that God knows all the classes and Gabriel only the sets. Gabriel presents to God a concept and asks for the corresponding class. That is, he asks for the class of everything that falls under that concept.

Let me use, as Fine does, the metaphor of classes as boxes. When Gabriel submits a concept, say that of not being the number 4, God opens that box which contains everything that is not 4. This is how Gabriel gets to know the complement of 4, which is a proper class.

But Gabriel can only give a concept that he already understands. And since, by assumption, he does not yet know the classes, he can only give concepts formulated in terms of the ordinary set-theoretic elementhood relation. So God and Gabriel go through all these concepts. Being 4, not being 4, being an ordinal, not being an ordinal, etc.

At the end, Gabriel has got to know a number of proper classes: the complement of 4, the class of all ordinals, the class of all sets that aren't an ordinal, etc. More precisely, what Gabriel has learned is that certain classes have such and such members. His concept of membership has extended. But he does not yet know classes that correspond to concepts formulated in terms of proper classes.

So God and Gabriel do a second round, in which Gabriel now is allowed to use his newly acquired knowledge of proper classes. This means, Gabriel now submits to God concepts that are formulated in terms of the extended membership relation. Again, God opens the corresponding boxes, and new classes are added to Gabriel's understanding of membership. They iterate these rounds and extend the membership relation step by step. Finally, the universe of classes is exhausted and every class has been added to the membership relation. At this final stage, Gabriel has fully understood what it means to be the member of a class.

## 3 Formal Construction (Draft)

This intuitive picture can be turned into the definition of a model of class-theory whose membership relation is obtained from ordinary set-theoretic elementhood in a stepwise manner analogous to Gabriel's exchange with God.[1]

God reveals the members of a class when Gabriel submits the corresponding concept. This idea is formalized by a function $\Delta$ that maps the urelemente (which represent the

[1]For details see [Fine, 2005, §3], [Speck, 2011, §1.2].

classes) to predicates.

**Definition 1.** $\Delta(c) = \phi_\alpha$ iff $\mu(c) = \langle \alpha, \beta \rangle$

The starting point of Fine's construction is a model of set theory with urelemente. These urelemente represent the classes before their members have been determined — first order objects without elements. At least, without members in the sense of the set-theoretic elementhood relation. These objects are *not* in the range of the membership relation.

**Definition 2.** $\mathfrak{M}_0 = \langle V_\kappa(C), e_0 \rangle$.

The membership relation $e_0$ is the ordinary set-theoretic relation such that $x\, e_0\, y$ if $x$ is an element of the set $y$. Therefore, no class yet is in the range of $e_0$. However, in terms of $e_0$ many concepts can be expressed that define proper classes. For example, the predicate $x \notin \{4\}$ defines the complement of 4, if '$\in$' is interpreted as $e_0$. Some urelement $c$, now, is mapped to this predicate ($\Delta(c)$ = '$x \notin \{4\}$'). This $c$ represents the complement of 4, and the membership relation is extendend by pairs $\langle x, c \rangle$, where $x$ satisfies '$x \notin 4$', as interpreted by $e_0$ ($x \in |$'$x \notin 4$'$|_0$).[2] The first *class*-membership relation $e_1$ is obtained by adding to the range of $e_0$ the urelemente $c$ such that $\Delta(c)$ defines a proper class if interpreted in the ground model of set theory, just as Gabriel first extends his understanding of membership to classes that he can define from his knowledge of set theory. Formally, $x\, e_1\, y$ if $x\, e_0\, y$ of $x \in |\Delta(y)|_0$. Generally, at any successor stage $\alpha$ a new membership relation $e_{\alpha+1}$ is defined whose range extends its predecessor $e_\alpha$ by urelemente $c$ such that $|\Delta(c)|_\alpha$ defines a proper class.[3]

**Definition 3.** For $c, d \in C$, $c \ll d$ iff $\mu(c) = \langle \alpha, \beta \rangle, \mu(d) = \langle \gamma, \delta \rangle$ and $\beta < \delta$, or $\beta = \delta$ and $\alpha < \gamma$

**Definition 4.** The ground model $\mathfrak{M}_0$ has been defined (def. 2). Given $\mathfrak{M}_\alpha$ let

$$\begin{aligned}\mathfrak{M}_{\alpha+1} =& \langle V_\kappa(C), e_{\alpha+1} \rangle \text{ where } x\, e_{\alpha+1}\, y \text{ iff } x\, e_\alpha\, y, \text{ or } x \in |\Delta(y)|_\alpha \\ & \text{and for any } \gamma \leqslant \alpha \text{ there is no } c \in C \text{ such that } c \ll y \text{ and } |\Delta(y)|_\alpha = |\Delta(c)|_\gamma. \\ \mathfrak{M}_\gamma =& \langle V_\kappa(C), e_\lambda \rangle \text{ with } e_\lambda = \bigcup_{\beta<\gamma} e_\beta, \text{ for limit ordinals } \gamma\end{aligned}$$

[2] Recall that a membership relation can be viewed as a collection of ordered pairs $\langle x, y \rangle$ such that $x$ is a member of $y$.

[3] At limit stages, the new membership relation is the union of all preceding ones.

In the following, I will talk of 'membership sequences' only in the sense of this definition, and will mean by 'membership relation' some $e_\alpha$ as it occurs in such a sequence of models $\mathfrak{M}_\alpha$.

**Proposition 5.** The classes of the models $\mathfrak{M}_\alpha$ are extensional. For any $c, d \in C$ and any $\alpha$, if $c, d$ are ascribed members $(\exists x(x\, e_\alpha\, c \wedge x\, e_\alpha\, d))$ then

$$\forall x(x\, e_\alpha\, c \leftrightarrow x\, e_\alpha\, d) \rightarrow c = d$$

Fine defines the *order* of a class as the stage where its members are revealed [p. 554]. Since $c$ enters the range of $e_{\alpha+1}$ just in case that $|\Delta(c)|_\alpha \neq \emptyset$, we can alternatively set

$$\text{order}(c) = min\{\alpha + 1 : |\Delta(c)|_\alpha \neq \emptyset\}$$

Thus, to use the picture again, the order of a class is the stage when the box has been opened and its content been determined.

The range of the membership relation increases strictly. More generally,

**Proposition 6.** for any membership sequence, the range of the membership relation increases monotonically, in the sense that for every $\alpha, \beta < \lambda$,

$$\text{If } \text{rn}(e_\alpha) \subseteq \text{rn}(e_\beta) \text{ then } \text{rn}(e_{\alpha+1}) \subseteq \text{rn}(e_{\beta+1})$$

As a consequence, the construction eventually closes off — there is a terminal stage $\lambda$ whose model $\mathfrak{M}_\lambda$ is a *fixed point.* Every class definable in $\mathfrak{M}_\lambda$ is already in the range of its membership relation $e_\lambda$.

The model $\mathfrak{M}_\lambda$ corresponds to the final round of God and Gabriel's dialogue (**??**), at the end of which Gabriel has fully understood class membership. How large this terminal ordinal $\lambda$ really is depends on how quickly the urelemente $C$ are used up. This again is a matter of which predicates $\phi(x)$ the classes are mapped to, and therefore depends on $\Delta$.

Fine, however, prefers to fix the terminal ordinal directly. For this, he introduces the notion of *class-inaccessibility.* $\lambda$ is class-inaccessible if there is no ordinal $\alpha < \lambda$ such that for any membership sequence $\mathfrak{M}_\alpha$ defines a well-ordering of order-type $\lambda$ (if there is such an ordinal $\alpha$, $\lambda$ is *accessible*).

# 4 Regularity

Eventually, Fine focuses on a specific family of models.

> I wish to propose the regular terminal models $M_\lambda$, for lambda class- inaccessible, as the intended models for the theory of classes. [Fine, 2005, p. 557]

A model is regular if it is defined in terms of a regular assignment $\Delta$. The regularity of $\Delta$, again, is defined in terms of the relation '... occurs in the predicate that $\Delta$ maps to ...', or just '... occurs in $\Delta$(...)' Now,

**Definition 7.** (Regularity) $\Delta$ is regular if for every $c \in C$, the relation '... occurs in $\Delta$(...)' is well-founded on the urelemente $C$.[4]

Although Fine does not make this connection explicit, his Regularity requirement follows from a general norm of real definition that he develops in the final section of his paper.

(Norm) Definitions must be well-founded. More precisely, the relation '... is used to define ...' is well-founded on the objects defined.

When applied to Fine's class-theory, (Norm) becomes his regularity requirement. First, notice that it is assignments $\Delta$ that fix how the classes are defined. Accordingly, in this special case (Norm) becomes a requirement on $\Delta$. For some such $\Delta$, now, the relation '... is used to define ...' is just the relation '... occurs in $\Delta$(...)'. Finally, the objects defined are the urelemente in $C$. In sum, $\Delta$ satisfies (Norm) if the relation '... occurs in $\Delta$(...)' is well-founded on $C$; that is, if and only if it is *regular*.

# 5 A Case for Irregular Assignments

## 5.1 Irregular Delta don't lead to ill-founded Class Definitions

Having described a novel model theory for impredicative classes (§3), Fine restricts his attention to *regular* models. He allows only regular assignments $\Delta$ to bijectively map urelemente onto predicates. Fine does so in order to rule out ill-founded definitions.

[4]Recall that a relation $R$ is well-founded on a set $A$ if every non-empty subset $B$ of $A$ has an element to which no $b \in B$ bears $R$.

And surely, if the urelemente in $C$ really are assigned formulae in a regular manner, none will ever represent a class of ill-founded definition. Regularity ensures well-founded class definitions, so to speak, by brute force. But the question remains, is regularity necessary?

Consider a $\Delta_i$ that maps $c \in C$ to '$x \neq d$' and $d \in C$ to '$x \neq c$'. By the extensionality of Fine's classes (proposition 5 above), '$x \neq d$' is equivalent to '$\forall y(y \in x \leftrightarrow y \in d)$'. Therefore, at any stage $\alpha$ where $d$ is not in the range of the membership relation, $|\Delta_i(c)|_\alpha$ is empty, such that $c$ is not added to the range of $e_\alpha$. In other words, $c$ will be ascribed members only when $x \in d$ is satisfied by some objects of $V_\kappa(C)$. However, for $d$ to enter $\mathrm{rn}(e_\beta)$ for $\beta \leqslant \alpha$, $c$ would have entered the range of the membership relation first — which contradicts the assumption that $\alpha < \beta$. The same reasoning applies to $\Delta_i(d)$; hence, neither $c$ nor $d$ ever enter the range. $c$ and $d$ do not come to represent classes; they remain what they are: urelemente. Generally, no urelement will enter the range of the membership relation if it is assigned a formula that presupposes an individual to have members that has not entered the range before.

**Proposition 8.** For every $\Delta$ and every $c \in C$, if $\exists\alpha$: $\mathrm{order}(c) = \alpha$ then $\forall d \in C$, if $\Delta(c) \models$'$x \in d$' then $\mathrm{order}(d) < \mathrm{order}(c)$.

What does this mean? On the one hand, it means that whenever we have an irregular assignment of urelemente to predicates, some urelemente will never be interpreted as classes. Irregular assignments $\Delta$ yield models of class theory *with urelemente.*

On the other hand, it means that although irregular assignments $\Delta_i$ allow for ill-founded dependence chains in Fine's sense, they do *not* lead to ill-founded class definitions. An ill-founded concept fails to define any class.

Let me sum up my case for irregular assignments $\Delta$. First, notice that the goal is merely to show that every *class* definition is well-founded. Further, recall that the urelemente are only interpreted as classes when they enter the range of the membership relation. To see why Regularity is an unnecessary restriction it now suffices to note that urelemente whose corresponding predicates involve terms not yet interpreted as classes simply do not enter the range of membership. Hence, these objects that correspond to ill-founded concepts *aren't classes.* Consequently, there are no ill-founded *class* definitions.

### 5.2 How Well-Founded Definitions are Ensured

Contrary to Fine's suggestions, irregular assignments do not give rise to ill-founded class definitions. This finding asks for closer investigation. If not regularity, what else ensures the classes of Fine's theory to be defined in their well-founded manner?

First, consider again Fine's example of an irregular assignment $\Delta_i$ such that $\Delta_i(c)$ is '$x \neq d$' and $\Delta_i(d)$ ='$x \neq c$'. If the urelemente were interpreted as classes all at once, this configuration would cause trouble. However, this approach is doomed anyway, as it would likewise require the Russell concept '$x \notin x$' to fix a class. and when he constructs his models, Fine carefully avoids such calamity. How does he do it? He defines his classes step by step. More precisely, the class membership relation is developed in stages; and an urelement is interpreted as a class at stage $\alpha + 1$ only if the predicate that it is assigned to is the first to carve out a certain region of the universe (definition 4 above}. In particular, $c$ enters the range of the membership relation $e_{\alpha+1}$ just in case $\Delta(c)$ is satisfied in the structure $\langle V_\kappa(C), e_\alpha \rangle$. But this cannot happen at any stage, as it requires $d$ to have been interpreted as a class before.

The reason why urelemente of ill-founded assignment are not interpreted as classes of ill-founded definition, is that at each stage of the construction, every class that comes to be defined, is defined in terms of earlier stages.

The assignment $\Delta$ of urelemente onto predicates is the backbone of Fine's construction. In order to rule out ill-founded class definitions he requires these functions to be regular. However, irregular assignments do not lead to ill-founded class definitions either. I conclude that Fine's regularity requirement is an unnecessary constraint on his model theory.

## 6 Groundedness

The preceding may seem a merely technical study. It is not.

In the remainder of this paper I will attempt to uncover the metaphysical principle that accounts for the success of Fine's model theory.

Moreover, it underlies the well-foundedness norm for definitions. Thus, it suggests an explanation why Fine came to impose his regularity constrain.

It is the principle of groundedness.

## 6.1 The Grounding Character of Fine's Model-Theory

As the previous section has shown, irregular assignments do not give rise to ill-founded class definitions. The reason is the classes are defined in stages, and any class is defined only if the defining formula is satisfied at the previous stage. This desirable feature of definition 4 is no happy accident or artefact of the formal set up. There is a general principle at work. The well-founded class definitions of the models $\mathfrak{M}_\alpha$ is just another chapter of a success story in 20th century logic. It is the story of *groundedness*.

Maybe the best known example of groundedness is the cumulative hierarchy of sets which provides axiomatic set theory with independent justification, *pace* Quine.

Herzberger first applied the groundedness idea to semantics (1970); but only Kripke (1975) gave the first grounded truth theory. Recently, Hannes Leitgeb proposed a different approach to grounded truth (2005).

Nonetheless, all these are cases of the following general pattern.[5] In a nutshell, some objects $S$ are grounded in a collection $G$ if you arrive at $S$ from $G$ by applying successively some operation $\gamma$ of the right, *grounding* kind. This operation and its iteration are two distinct aspects of the notion of groundedness. In order to spell it out I will consider these aspects separately, explaining first the grounding character of $\gamma$, and then say something more about its iteration.

You may think of $\gamma(G)$ as a construction from $G$, but only metaphorically. The proposed account of groundedness is meant to be thoroughly realist. Groundedness, as I think of it, is an objective property. Objects are grounded in virtue of how the world is like, independently of our constructive abilities.[6] Consequently, there are no limits as to how $\gamma(G)$ is computed but for one crucial constraint: the only input is $G$.

However, the interesting collections of objects grounded in $G$ are not obtained in a single step. Instead, the operation $\gamma$ is iterated. Once the objects $\gamma(G)$ are obtained, they may be used themselves as input for $\gamma$. Thus, another collection $\gamma(\gamma(G))$ is generated, which again is grounded in $G$; and so on. Notice that since the grounding operation is iterated, some conception of ordinal number is built into the notion of groundedness. This is the 'step-by-step' aspect of groundedness that I plays a central role in Fine's construction.

I do not think that while $\gamma$ is iterated, the collections necessarily become bigger and

[5]Elsewhere, I develop a formal theory of groundedness [Speck, 2011].

[6]For the sake of readability, I will nonetheless make frequent use of construction talk; this will always be merely metaphorical.

bigger, or richer and richer. $\gamma(X)$ may well be just a fragment of $X$. Nonetheless, every new collection obtained from applying $\gamma$ is *grounded* in the starting point $G$. Especially, $S$ being grounded in $G$ does not mean that no new collection $\gamma(S) \neq S$ could be obtained from it (groundedness does not imply being a fixed point).

Since groundedness does not depend on any subject to carry out all the iterations, there are no constraints as to how many times the operation is applied. Therefore, the iteration of $\gamma$ is continued beyond limit stages and the notion of ordinal numbers at work in groundedness is fully transfinite.

In sum, a collection $S$ is grounded in $G$ if there is an operation $\gamma$ such that, if you start from $G$ and iterate $\gamma$ transfinitely often, whereby at each stage you only use what the previous stage has given, you arrive at $S$.

Given this general account of groundedness, it is not difficult to see that Fine's membership relations $e_\alpha$ are *grounded* in ordinary set-theoretic elementhood. Let $S$ be any class membership relation, that is, a collection of pairs $\langle x, y\rangle$, where $x$ is a member of $y$. The grounding operation $\gamma$ simply maps class membership relations $e_\alpha$ to $e_{\alpha+1}$ Formally, for any collection $R$

$$\gamma_F(R) = \begin{cases} e_{\alpha+1} & \text{if } R = e_\alpha \text{ for some } \alpha \\ R & \text{otherwise} \end{cases}$$

Finally, let $G$ be the elementhood relation of ordinal set theory — the collection of pairs $\langle x, y\rangle$, now for $x$ being an element of the *set* $y$.

If starting from $G$, $\gamma_F$ is iterated along the ordinals, one arrives at $S$. This is just what makes $S$ a membership relation in the sense of definition 4. Moreover, $\gamma_F$ is an operation of the right, grounding kind. It captures the definition 4 from section 3, and there, $e_{\alpha+1}$ is defined only in terms of $e_\alpha$. Thus, $\gamma_F(R)$ is obtained solely on the basis of $R$. At each stage of the construction of $S$ from $G$, nothing is presumed than what is given from the previous stage. Class membership is grounded in set elementhood.

The previous section, showed that for *irregular* models $\mathfrak{M}_\alpha$, too, the classes as represented by urelemente in the range of $e_\alpha$ have well-founded definitions. Technically, this is because the range of the membership relation is extended in a specific step by step manner.

Now, the philosophical significance of this explanation becomes apparent. Regularity is not needed to ensure well-founded class definitions since class membership is *grounded* in set-theoretic elementhood.

### 6.2 Well-Foundedness as a Groundedness Norm (Draft)

But what about the general well-foundedness norm on definitions?

Unfortunately, Fine does not argue for the well-foundedness requirement. He does not explain why the '... defined in terms of ...' relation should be well-founded.

But I think you can make the following case. My starting point is the *eliminability* criterion for definitions: In other words, for every definition there is to be a way of tracing it back to primitives.

On a closer look, however, it will turn out that this is just the case if the definition is grounded in the primitves, grounded in the very sense in which class membership relations are grounded in set membership.

Using the equivalent dependence description of groundedness (Yablo): S grounded in G if there is a G-dependence relation R such that every R-path starting from S is finite.

The relation '... is defined in terms of ...' is well-founded on objects D, given some primitives P, iff Every objects in D is grounded in P.

That well-foundedness is a groundedness idea can also be seen in that Fine explains the well-foundedness requirement as that any definition should 'bottom out' [Fine, 2005, p. 568].

By way of motivation, you can also argue that historically, well-foundedness ideas have been developed in close connection with the iterative conception of set (Mirimanoff, Zermelo...).

## References

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