Putting *Abfolge* to Use: An Iterative Conception of Classes

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Section 1

Introduction

Idea-Extensions and Russell's Paradox

- Bolzano can be read as endorsing the following principle (Berg 1962, 71):

  Every non-empty idea Φ *has an extension* $\hat{x}\Phi x$ *such that for every object a, a is contained in* $\hat{x}\Phi x$ *just in case a has* Φ.

1. Socrates is mortal. But, the extension of the idea of mortality is not.
2. Hence, the idea of being an extension that is not contained in itself is non-empty.
3. Call its extension r. It is contained in itself just in case it is not. Contradiction.

Contents

Section 2

Two Ideas of Collection

Combination and Definition

Combination We collect some things by a sequence, possibly uncountable, of independent decisions whether a given object belongs to them or not.

- A *combined* collection we call a *set*.
  - $\{x, y, ...\}$
  - x is an element of the set y: $x \in y$

Definition We collect some things by means of a *condition*, which exactly they satisfy.

- A *defined* collection we call a *class*, or a *concept-extension*.
  - $\{x : \Phi x\}$
  - x is a member of the class y: $x \eta y$

Naive Comprehension and Paradox 1

NCC Let Φ be any condition. There is a class $\{x : \Phi x\}$ such that for every object a,
$a\eta\{x : \Phi(x)\}$ gdw. $\Phi(a)$

- $\{x : \neg x\eta x\}\eta\{x : \neg x\eta x\}$ iff $\neg\{x : \rho(x)\}\eta\{x : \rho(x)\}$
- Is (Definition) bankrupt? No!

Naive Comprehension and Paradox 2

- There's also a naive notion of *set*.

  NSC Let xx be some things. There is a set $\{xx\}$ such that for every object a,
  $a \eta xx$ iff a is among the xx.

- It, too, leads to paradox:
- Let the rr be the sets that don't contain themselves.

Naive Comprehension and Paradox 3

- The *naive* notion of set was *replaced* by the *iterative conception* of set.
- I will develop (Definition) into an *iterative conception of class*.

Section 3

Iterative Conception of Set

Stages

- A set is *constituted* from its elements: It *presupposes* its elements.
- The sets come in *stages*:
  1. $\varnothing$ presupposes nothing.
  2. $\{\varnothing\}$, as $\varnothing$ is given.

  ...

  α. Sets of sets of stage $< \alpha$.

Definition (Dependence) We say that x depends on y if y stands in the transitive closure of $\in$ to x.

Definition Let x be a set.
$\text{rank}(x) = sup\{rank(y) : x \text{ depends on } y\} + 1$

- We get:

SC Let α be a rank, and xx some objects *of rank* $< \alpha$. There is a set $\{xx\}$ such that for every object a $a \in \{xx\}$ iff a is among the xx.

Constituency

"The *xx constitute* $\{xx\}$"

- What does this mean?

1. The Kingdom of Norway is constituted from the Norwegians.
2. The meaning of '+' is constituted from the usage of this symbol.
3. This quadrangle is constituted from these two triangles.

Existence If the *yy* constitute x then x and the *yy* exist.

Uniqueness If the *yy* constitute x, and the *zz* constitute x, then the *yy* are the *zz*. Similarly, if the *yy* constitute x, and the *yy* constitute y, then $x = y$.

Non-Circularity There is no sequence of objects $x_1, \ldots, x_n$ such that for every $i < n$, x_{i+1} is among the objects which constitute x_i, and $x_1 = x_n$.

Section 4

Iterative Conception of Proper Classes

Truths

- (Definition) motivates a change of perspective:
- Away from *objects*, to *true propositions*.
- All I assume of a proposition is:
  - Abstract
  - Structured and finely individuated
- I'll write '[A]' for the proposition that A.
  - I'll call $[x \in y]$ and logical functions of it '$\in$-propositions',
  - $[x\eta y]$ and logical functions an 'η-proposition'.

From Definition to Truths

- Which classes are there?
- (Definition): If Φa and $[\Phi a]$ *safe* then $a\eta\{x : \Phi(x)\}$, hence $\exists y(a\eta y)$.
- Which propositions hold and are safe?

Predicative Classes

- Which propositions hold and are safe?
- All truths of set-theory!
- There are the classes $\{x : \Phi x\}$, Φ set-theoretic.
- *Old News*: We already know predicative class theory.

Impredicative Classes

- Which η-propositions hold and are safe?
- "Does $[a\eta\{x : \Phi x\}]$ hold?" $\Rightarrow$ "Does $[\Phi a]$ hold?"
- $[\Phi a]$ may itself be an η-proposition.
- η-propositions depend on η-propositions.
- As *Platonists*, how do we explicate this?

Bolzanian Grounding 1

- Propositions stand in the relation of ground and consequence ('$\lhd$').

  Grounding is constituency for the definitional idea.

1. [The angles of a triangle add up to 180 degrees] $\lhd$ [The angles of a quadrangle add up to 360 degrees]. (WL §162)
2. [God is perfect] $\lhd$ [The actual world is the best of all worlds]. (WL §201)

- This relation is *not* epistemic (WL §198), *not* causal (WL §201) and *stricter* than logical consequence (WL §200).

Bolzanian Grounding 2

Faktivität If $A_0, A_1, \ldots \lhd B_0, B_1, \ldots$, then $A_0, A_1, ..., B_0, B_1 \ldots$. (WL §203)

Uniqueness If $\Gamma \lhd \Delta$ and E $\lhd \Delta$ then $\Gamma = E$. Similarly, if $\Gamma \lhd \Delta$ and $\Gamma \lhd E$ then $\Delta = E$. (WL §206)

Non-Circularity There is no chain $\Gamma_0, ..., \Gamma_n$ such that for every $i < n$, Γ_i ground Γ_{i+1} and there's an A that is among the Γ_0 as well as among the Γ_n. (WL §§204, 218)

Proper Classes and Grounding

- Why $a \eta \{x : \Phi x\}$?
- Because a is a Φ.
- $[\Phi a] \lhd [a \eta \{x : \Phi x\}]$
- I remain neutral as to how grounding interacts with the connectives.

Putting Grounding to Use

- Bolzanian grounding allows to to *order* the eta-propositions without compromising our *platonism*.

Definition (Dependence) We say that A *depend* on B if B is among some propositions that stand in the transitive closure of grounding to A. (WL §217)

Iterative Conception of Proper Classes 1

(Basic Truths) For every logical or set-theoretic truth A there are no eta-propositions Γ such that $\Gamma \lhd A$.

Corollary Dependence is well-founded on the $\in$- and η-propositions.

Definition (Rank) Let A be a truth.
rank(A)=sup{rank(B) : A depends on B}+1

- Set-theoretic truths have rank 0.
- Truths $[a\eta\{x : \Phi(x)\}]$, $\{x : \Phi x\}$ *predicative*, have rank 1.

Iterative Conception of Proper Classes 2

Definition (Grounded Truths) A is *grounded* iff it has a rank.

CC For every condition Φ and every object a. If [Φa] is grounded, then:
$a\eta\{x : \Phi(x)\}$ just in case $\Phi(a)$

Examples

- (Universal Class) $[\{x : x = x\} = \{x : x = x\}]$ does *not* depend on any η-proposition. It is a *grounded* truth.

$$\{x : x = x\}\eta\{x : x = x\} \text{ iff } \{x : x = x\} = \{x : x = x\}$$

- (Russell) Assume

$$\text{(R)} \quad [\neg\{x : \neg x\eta x\}\eta\{x : \neg x\eta x\}]$$

has rank α. (R) depends on itself. Hence (R) must have rank $\beta < \alpha$. Contradiction.
The Russell-proposition is *ungrounded*.

Section 5

Conclusion

Wrapping Up

- A new response to the class-theoretic paradoxes.

1. The iterative conception of *set* bases on the primitive relation of *constituency*.
2. I take (Definition) seriously. It's about *facts*, not objects.

- Bolzanian grounding is constituency for classes.
- We obtain a *cumulative hierarchy* of class-theoretic truths.

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