# The Groundedness of Fine's Class Membership

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Kit Fine (2005) shows the consistency of an impredicative class theory. He constructs models that extend the cumulative hierarchy; their theories, however, not only imply the existence of a universal class but are closed under the Boolean operations (union, complementation etc.) in general.

If this was not sufficient to attract attention, Fine also bases his construction on a novel analysis of the set-theoretic paradoxes. In the present paper, I elaborate on his ideas. Fine's membership relation, I argue, is *grounded* in ordinary set-theoretic elementhood.

The paper is structured as follows. First, I explain Fine's ideas in informal terms (§1.1). Section 1.2 presents his model-theoretic construction as well as the class-theoretic axioms that it validates. Along the way, I will fill minor gaps in Fine's own presentation (§1.2.3). Then, I discuss in more detail Fine's underlying account of well-founded definitions (§2.1). I argue that it needs be supplemented by an account of grounded membership. I outline the relevant notion of groundedness and develop a formal definition (§3). Finally, I show that Fine's class-membership is indeed grounded in set-theoretic elementhood in the precise sense of the definition proposed.

## 1. Fine's Theory of Classes

### 1.1. Philosophical Motivation

Fine's starting point is a shift in perspective: it is not the universe that is extended — all objects are given at the outset of the construction. Instead, the membership relation is developed in a step by step manner, whereby the members of more and more classes are 'revealed'.

To motivate this unusual order of definition, Fine invites the reader to think of classes as boxes [Fine, 2005, p. 548]. As long as the lid is closed, the members are hidden. But the containers may be opened, or made transparent, and their members be identified. In this picture, it seems natural that the classes are given first, if only closed, and the membership relation is developed later.

As Fine does, let me first describe his construction by means of a little story. Imagine a dialogue between God and archangel Gabriel. Both Gabriel and God know all proper classes but only God knows their members. Now assume that they engage in a question-answer game in the course of which God opens more and more of the boxes, that stand for to the proper classes.

It is how God opens the boxes that illustrates a central feature of Fine's construction. She does not take a box and opens it — rather, Gabriel presents to God some things and then God opens the box that they are in. But how does Gabriel identify these things? Since God and Gabriel are just about to determine its content, these things cannot be referred to as what is in the box. Instead, Gabriel uses a concept[1] to specify the collection of things that it is true of.

For example, when Gabriel comes up with a concept, say that of not being the number 4, God opens one of the boxes, namely that box which contains everything that is not 4. This is how Gabriel gets to know the complement of 4, which is a proper class.

In effect, the classes of Fine's theory coincide with predicate-extensions. The notion of class that drives his construction is what at the end of the paper he calls the 'logical conception' [Fine, 2005, p. 568]. That is, Fine's classes derive from concepts: the members of any class are just those which satisfy a certain condition.

In view of this, the question of paradox naturally becomes pressing. As the failure of naive set theory and Frege's *Grundgesetze* show, concepts must not carelessly be mapped into the first order domain. However, Fine's proposes a novel and elegant solution.

Traditionally, the class-theoretic antinomies have been blamed on naive comprehension. Fine suggests a different analysis. Below, I will discuss his motivation in more detail (§2.1). For the time being it suffices to point out that Fine's treatment of the paradoxes is based on the new perspective he adopts, the 'reversal in the roles of the

[1] Fine himself speaks of 'conditions'.

predicate of membership and the ontology of sets' (op. cit.). If the membership relation is not any longer assumed to be unequivocal and given at the outset but on the contrary seen as the result of a step-by-step construction, the heedless use of *naive membership* may be seen as the culprit.

Using Fine's picture again, in the beginning Gabriel can only present to God concepts that he already understands. And since, by assumption, he does not yet know the proper classes, he can only give set-theoretic concepts, that is, predicates formulated in terms of the ordinary set-theoretic elementhood relation. So God and Gabriel go through all these concepts: Being 4, not being 4, being an ordinal, not being an ordinal, and so on. At the end, Gabriel has got to know a number of proper classes, the complement of 4, the class of all ordinals, the class of all sets that aren't an ordinal. More precisely, what Gabriel has got to know is that certain classes have such and such members. His concept of membership has extended.

This new diagnosis also suggests a natural repair. If the membership relation is developed in stages then the phrase '$x$ is a member of $y$' expresses different concepts at different stages, as do complex formulae built up from it. Moreover, this re-interpretation proceeds in a way such that formulae which on their usual, *naive* interpretation lead to paradox now give rise to concepts that can coherently be taken to have definite extensions. These are the classes of Fine's theory.

## 1.2. Technical Implementation

Fine does not leave it at the philosophical motivation as described in the preceding section. He develops his theory of classes in more formal terms — he sketches the construction of models. Nonetheless, I consider it worthwhile to set things out even more explicitly. It will allow me to clear up certain ambiguities in Fine's presentation.

### 1.2.1. The Ground Model: Set Theory with Urelemente

Since Fine's class theory is supposed to extend the set-theoretic universe $\mathbf{V}$ and imply, for instance, a class of all sets, Fine cannot literally define a model for it. What he can do instead, however, is to define a *set*-model. Its existence he can prove in ordinary ZFC extended by the assumption of an inaccessible cardinal. In addition, though, this construction is also a model in the scientist's sense. From truth in the set-model we can generalize to truth in the real world of classes. Thus, Fine's set-theoretic construction serves to motivate a theory that vastly exceeds standard set theory.

Fine starts from a set of urelemente $C$. On the intended interpretation, these urelemente are the proper classes. Fine aims for a class theory closed under complementation, hence for every set there needs to be a class of its non-elements. Thus, there must be at least as many urelemente as sets. Since Fine works in ZFC+'There is an inaccessible $\kappa$', the size of $C$ is taken to be this $\kappa$.

The universe of classes now is modelled by the cumulative hierarchy on the basis of $C$. However, this hierarchy must not, as usual, be based on the power-set operation. If it was then already at the first stage there would be, contrary to Fine's intention, many more sets than classes. The $2^\kappa$-many sets $\mathcal{P}(C)$ could never each have its own complement class in $C$. Fortunately, this difficulty is avoided as follows. At any stage $\alpha + 1$, instead of the full power set $\mathcal{P}(V_\alpha(C))$ we confine ourselves with the subsets of size less than $\kappa$ ($\mathcal{P}^\kappa(V_\alpha(C))$). Then, $V_\kappa(C)$ has cardinality $\kappa$ itself.[2]

$V_\kappa(C)$ now is the domain of Fine's models. It will remain unchanged all through the construction. On it, though, larger and larger membership relations $e$ are defined. The starting point is ordinary set-theoretic elementhood $e_0$. It gives rise to a first model,

**Definition 1.** $\mathfrak{M}_0 = \langle V_\kappa(C), e_0 \rangle$.

The range of $e_0$ contains only the pure sets of $V_\kappa$. At this first stage, the elements of $C$ are not yet in the range of the membership relation.

### 1.2.2. Mapping Urelemente to Formulae (1)

However, many predicates of the language of set theory define proper classes, among which $\ulcorner x = x \urcorner$, or $\ulcorner \exists z(z \neq \varnothing \wedge \subseteq x \forall y \in z(y \cap z \neq \varnothing) \urcorner$ ('$x$ is ill-founded'). The core of Fine's construction are functions $\Delta$ that map the urelemente into these conditions [Fine, 2005, p. 553]. In fact, Fine allows for conditions formulated in an $\mathcal{L}_{\kappa\kappa}$ extension of the first order language of set-theory that also contains a constant for every urelement [p. 551]. Interpreted in $\mathfrak{M}_0$, such a condition $\phi$ defines an extension $|\phi|_0 \subseteq V_\kappa(C)$. Thus, a new membership relation $e_1$ can be defined whose range now covers proper classes in $C$, too.

$$x\, e_1\, y \text{ iff } x\, e_0\, y \text{ or } x \in |\Delta(y)|_0$$

[2] $|\mathcal{P}^{<\kappa} V_\alpha(C)| = \kappa^{<\kappa}$ which on the assumption of $\kappa$'s inaccessibility is just $\kappa$.

$e_1$ includes the set-theoretic elementhood relation (left disjunct). In addition, though, its range now also contains urelemente. On the intended interpretation, the members of some classes have been revealed. Formally, if a class $c$ is mapped to a condition $\phi(x)$ (i.e. $\Delta(c) = \phi(x)$) and this predicate, according to $\mathfrak{M}_0$, is true of some objects, then $y\, e_1\, c$ if and only if $\phi$, according to $\mathfrak{M}_0$, is true of $y$.

For example, some $c$ will be mapped to the formula '$Sx$' ($\Delta(c) =$'$Sx$'). If this formula is interpreted in the model $\mathfrak{M}_0$ then it defines a non-empty subset of the domain $V_\kappa(C)$, in fact quite a large one, namely $V_\kappa(C) \backslash C$. Therefore, $c$ is in the range of $e_1$, and '$x \in c$' will be true in $\mathfrak{M}_1$ for every set $x$. Thus, $c$ represents the class of all sets.

Other 'boxes', however, remain opaque. There are classes whose extension cannot be expressed in terms of set-theoretic elementhood. In the model-theoretic construction, this is reflected by the fact that $\Delta$ maps some urelemente to formulae which do not 'deliver' if interpreted in $\mathfrak{M}_0$ — there are no objects that they are true of. One example is being in the complement of the ordinal 4.[3] The predicate

$$\exists y(x \in y \wedge \forall u(u \in y \leftrightarrow u \notin 4)) \tag{1}$$

has an empty extension if interpreted in $\mathfrak{M}_0$. If $\Delta$ maps some urelement to the condition (1), that is, if there is to be the complement of 4, the range of the new membership relation $e_1$ cannot yet exhaust $C$.

The members of some more classes will only be revealed in the next step, when a new membership relation is defined in terms of the function $|\Delta(c)|_1$. If this procedure is iterated transfinitely many times, it gives rise to a sequence of models.

$$\mathfrak{M}_{\alpha+1} = \langle V_\kappa(C), e_{\alpha+1}\rangle \text{ where } x\, e_{\alpha+1}\, y \text{ iff } x\, e_\alpha\, y \text{ or } x \in |\Delta(y)|_\alpha$$
$$\mathfrak{M}_\gamma = \langle V_\kappa(C), e_\lambda\rangle \text{ with } e_\lambda = \bigcup_{\beta<\gamma} e_\beta, \text{ for limit ordinals } \gamma$$

For the definition of class theories, Fine proposes to focus on the subclass of *regular* membership sequences [Fine, 2005, p. 554]. Here, the definition of regularity is based on the following, simple notion of dependence. Given a mapping $\Delta$ as introduced above, $c$ $\Delta$-depends on $d$ iff $\Delta(c)$ contains a constant denoting $d$. For $x \in C$, let $\mathcal{D}(x)$ be the smallest set that contains every $y$ such that $x$ $\Delta$-depends on $y$. Now, $\Delta$ is *regular* iff for every $x \in C$, $\Delta$-dependence is well-founded on $\mathcal{D}(x)$. A model is called 'regular' if

[3] Clearly, the domain of $(e_\lambda)$ cannot exceed the universe.

it is the terminal model of a membership sequence induced by a regular $\Delta$. Although Fine does not do so himself, the membership of a regular model may naturally be called 'regular', too.

According to Fine, regular models have two interesting properties [Fine, 2005, pp. 554n.]. First, regular membership is preserved under permutation of the urelemente not in its range. Any permutation on $C \backslash rn(e_\alpha)$ can be extended to an *automorphism* on the model $M_\alpha$. Intuitively, Gabriel is not able to distinguish between boxes that God has not yet opened, so that She may swap these without altering the procedure of their dialogue. Secondly, once the size of the universe has been fixed, the length of a regular membership sequence determines its terminal model up to *isomorphism*. Regular models $\mathfrak{M}_\alpha$ cannot distinguish between co-extensional urelemente.

Understanding the reason for this requires to go into more details. On the way, however, it will turn out that Fine's definitions need some modification. When these difficulties have been cleared up the specific properties of regular membership sequences will become clearer, too.

### 1.2.3. Indeterminate Membership

Fine defines the *order* of a class as the stage where its members are revealed [p. 554]. Since $c$ enters the range of $e_{\alpha+1}$ just in case that $|\Delta(c)|_\alpha \neq \varnothing$, we can alternatively set

$$\text{order}(c) = min\{\alpha + 1 : |\Delta(c)|_\alpha \neq \varnothing\}$$

Thus, to use the picture again, the order of a class is the stage when the box has been opened and its content been determined. Clearly, this interpretation of $|\Delta(c)|_\alpha$ (for $\alpha$ =order($c$)) as the members of $c$ makes sense only if once an urelement has been mapped to an extension of $V_\kappa(C)$, this extension does not change at higher stages.

Unfortunately, the construction as described so far does not provide the urelemente with unique extensions. There are formulae $\phi$ and stages $\alpha$ such that $\varnothing \neq |\phi|_\alpha \subsetneq |\phi|_{\alpha+1}$. Thus, on Fine's account, there will be a $c$ such that at different stages, different members are ascribed to $c$.

An example is the formula '$x$ is membered'.

$$\exists u(u \in x) \tag{2}$$

At the outset of the construction, when '$\in$' is interpreted as the ordinary set-theoretic elementhood relation, 2 is true of all and only the pure sets, i.e. $|(2)|_0 = V_\kappa(C)\backslash C$.

Already at the next stage, though, some urelemente have been added to the range of the membership relation such that (2) is now true not only of the sets but also of some of these. Formally, $|(2)|_1 = V_\kappa(C)\backslash(C\backslash \mathrm{rn}(e_1)$. In general, for any stage $\alpha$ $|(2)|_\alpha = V_\kappa(C)\backslash(C\backslash \mathrm{rn}(e_\alpha) \subsetneq |(2)|_{\alpha+1}$. For all these $\lambda$ many different extensions, however, there is just one urelement $c$ such that $\Delta(c) = (2)$; Fine explicitly wants $\Delta$ to be one-one [Fine, 2005, p. 553]. What, now, are the members of $c$? Fine's construction as he describes it does not determine the extension for all of its classes.

To show why this is a direct consequence of how $\Delta$ is defined, let me picture Fine's construction by a two-dimensional diagram (see figure 1). The vertical axis corresponds to the increasing membership relation and the horizontal lists the $\kappa$ many formulae. The result is a two-dimensional table mapping formulae to their extensions for increasing interpretations of the relation symbol '$\in$'.

On Fine's account, $\Delta$ maps the urelemente one-one to formulae [Fine, 2005, p. 553]. In consequence, every urelement corresponds to a column of the table. Therefore, as soon as one formula is mapped to more than one non-empty extension, there are more non-empty fields in the table than classes. This picture shows why Fine's construction must *undergenerate*: it fails to provide enough classes for all the $\kappa \times \lambda$ many extensions.

Fortunately, this way of looking at the problem already suggests a solution. If you wish to retain Fine's basic idea of a step-by-step reinterpretation of the membership relation as well as continue interpreting the urelements as concept-extensions, then you must no longer map the urelemente to formulae but to pairs of one formula and one stage. In other words, an urelement no longer corresponds to a *column* of the table, but to one of its *cells*. In the next section I will suggest a way to spell out this intuitive idea.

### 1.2.4. Mapping Urelemente to Formulae (2)

First, using some ordinal enumeration of the urelemente, and encoding of pairs, define a bijection $\mu : C \mapsto \kappa \times \lambda$. Figuratively speaking, $\mu$ maps every urelement to a cell of figure 1, represented by a pair of two ordinals. On this basis, enumerating the formulae according to their lexicographical order, define

**Definition 2.** $\Xi(c) = \phi_\alpha$ iff $\mu(c) = \langle \alpha, \beta \rangle$

Importantly, and this is how it differs from Fine's $\Delta$, the function $\Xi$ is not bijective. Instead, for every formula $\phi$ there are $|\lambda|$ many urelemente $c$ such that $\Xi(\phi) = c$. Despite

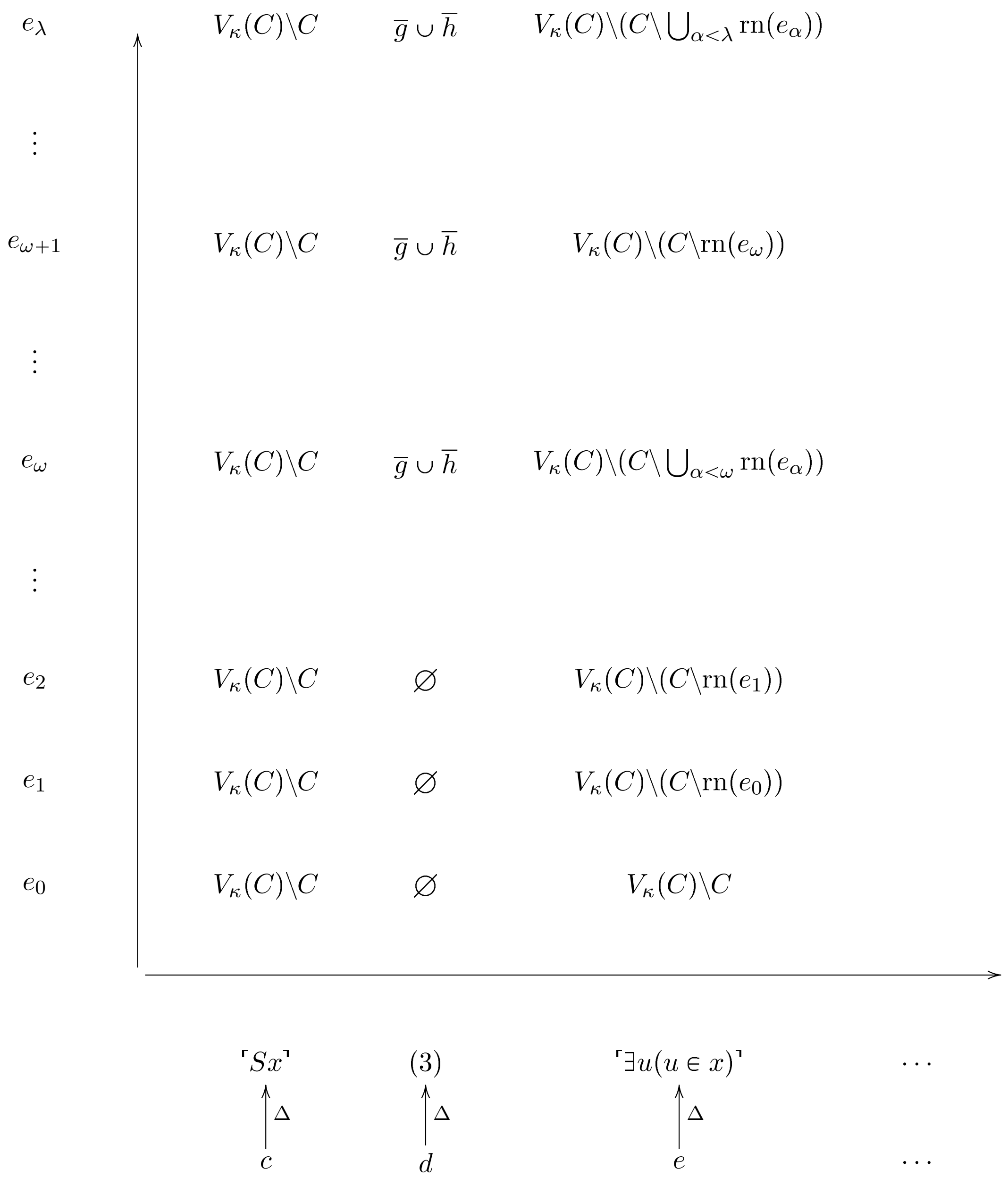


Figure 1: Fine's $\Delta$: Formulae and membership relations

this adjustment, the new mappings provide a smooth replacement of the original, unsatisfactory $\Delta$. Especially, Fine's notion of regularity (p. 5) can equally well be applied to the mappings $\Xi$.

At first, this modification seems already to have solved the problem of undergeneration. The increasing extensions of (2) now each make up a separate class. Generally, the bijectivity of $\mu$ ensures that

**Fact.** For any stage $\alpha$ and any formula $\phi_\beta$ there is a $c \in C$ such that for any $x$

$$x\, e_{\alpha+1}\, c \text{ iff } x \in |\phi_\beta|_\alpha$$

However, $\Xi$ gives rise to a new problem. For many formulae, their extensions remain constant from a certain stage on, for example the following ('$x$ is in the complement of $g$ or $h$').

$$\exists y(x \in y \wedge \forall u(u \in y \leftrightarrow u \notin g \vee u \notin h) \tag{3}$$

Assume that at the first stage (i.e. in the model $\mathfrak{M}_0$) none of the two classes $g$ and $h$ are revealed. This means, $|\Xi(g)|_0 = |\Xi(h)|_0 = \varnothing$. Therefore, both '$x \notin g$' and '$x \notin h$' are true of every object in the domain of $\mathfrak{M}_1$, i.e. $|x \notin g|_1 = |x \notin h|_1 = V_\kappa(C)$. But there is no object in the range of $e_1$ (which interprets '$\in$' in $\mathfrak{M}_1$) that is co-extensive with the $V_\kappa(C)$ ($\kappa$ is inaccessible). Hence, there is no witness for the existential quantification in (3) — interpreted in $\mathfrak{M}_1$, (3) is *false* of every $x$. For this reason, $|(3)|_1 = \varnothing$.

But assume that at stage 1, at least $g$ (but not $h$) is mapped to some set of objects such that $|\Xi(g)|_1 \neq \varnothing$. At the subsequent, second stage of the construction, '$x \notin g$' therefore is no longer vacuously true of everything. The formula '$\exists y(x \in y \wedge \forall z(z \in y \leftrightarrow z \notin g))$' therefore will have a non-empty interpretation in $\mathfrak{M}_2$. However, $|x \notin g|_2$ still is $V_\kappa(C)$ such that (3) is false in $\mathfrak{M}_2$, too. Therefore, (3) still has an empty interpretation in $\mathfrak{M}_2$: $|(3)|_2 = |(3)|_1 = \varnothing$. Only when both $\Xi(g)$ and $\Xi(h)$ have been mapped to non-empty interpretations, say at the third stage, '$x \notin g \vee x \notin h$' is false of some objects in the universe and (3) again true in $\mathfrak{M}_3$. In this case, however, the extension of (3) has been fixed also for any stage $\alpha > 3$.

This example shows that modified function $\Xi$ maps different urelemente to the same extension. Whereas $\Delta$ was not able to reflect differences, $\Xi$ now *overgenerates*. However, this difficulty can be resolved if the definition of membership is carefully modified, as I will explain in the next section.

### 1.2.5. Restricted Membership

The modification I would like to propose can also be motivated from Fine's heavenly dialogue, or from a modest development of his story. When Gabriel has submitted a condition, and God opened a box that contains just those objects of which the predicate is true, She commands Gabriel to look back at all the boxes they have opened so far. She lets him check if any of these contains the same objects as the one just opened. If so, God closes it again. Only when Gabriel has done so, may he continue with the next condition. Thus, God ensures that at the end of their game, no two open boxes have the same content.

Let me now formulate this idea within the framework of Fine's set-theoretic models. More precisely, I will add to Fine's definition of $e_{\beta+1}$ a constraint that corresponds to Gabriel's checking all previously opened boxes.

First, notice that the function $\mu$ induces a natural ordering of the urelemente, when the pairs of ordinals are arranged in the reverse lexicographical order.

**Definition 3.** For $c, d \in C$, $c \ll d$ iff $\mu(c) = \langle \alpha, \beta \rangle, \mu(d) = \langle \gamma, \delta \rangle$ and $\beta < \delta$, or $\beta = \delta$ and $\alpha < \gamma$

By means of the relation '$\ll$' we can express that some boxes are opened earlier than others. Thus, it allows me to sharpen the idea of looking back at the boxes opened so far, and take a first shot at the condition I wish to add to Fine' construction. An urelement $d$ that we have just for the first time mapped to a collection of objects is added to the range of the membership relation only if there is no $c \ll d$ of the same extension.

Due to the definition of '$\ll$', the condition 'there is no $c$: $c \ll d$' excludes those urelemente $c$ that have been assigned the same extension at some earlier stage of the construction (formally, $\text{order}(c) < \text{order}(d)$). Nonetheless, these stages are again referred to when we compare extensions $|\Xi(c)|_\alpha$ which are functions of $\Xi(c)$ *and* some stage $\alpha$. Therefore, to fully formalize the idea intended we also need to quantify over the stages $\alpha$.

In sum, I propose the following definition of models $\mathfrak{M}_\alpha$.

**Definition 4.** The ground model $\mathfrak{M}_0$ has been defined (def. 1). Given $\mathfrak{M}_\alpha$ let

$$\begin{aligned}\mathfrak{M}_{\alpha+1} =& \langle V_\kappa(C), e_{\alpha+1}\rangle \text{ where } x\, e_{\alpha+1}\, y \text{ iff } x\, e_\alpha\, y, \text{ or } x \in |\Xi(y)|_\alpha \\ & \text{and for any } \gamma \leqslant \alpha \text{ there is no } c \in C \text{ such that } c \ll y \text{ and } |\Xi(y)|_\alpha = |\Xi(c)|_\gamma. \\ \mathfrak{M}_\gamma =& \langle V_\kappa(C), e_\lambda\rangle \text{ with } e_\lambda = \bigcup_{\beta<\gamma} e_\beta, \text{ for limit ordinals } \gamma\end{aligned}$$

Henceforth, I will use the expression 'membership sequence' only in the sense of this definition and will mean by 'membership relation' some $e_\alpha$ as it occurs in such a sequence of models $\mathfrak{M}_\alpha$.

This slight modification of Fine's construction solves the problems of the original proposal. On one hand, the use of $\Xi$ ensures that each class is ascribed a definite membership (see proposition 1.2.4 above). Especially, the *order* (p. 6 above) of any $c$ is well-defined.

**Definition 5.** $\text{order}(c) = \mathit{min}\{\alpha : c \in \text{rn}(e_\alpha)\}$

On the other hand, the restriction now imposed on the definition of $e_{\beta+1}$ rules out that two different urelemente are assigned the same collection of objects. To consider the example from above, at the third stage, some urelement $u$ such that $\Xi(u) = 3$ is mapped to the union of the complements of $c$ and $d$. From now on, any urelement will only be added to the range of the membership relation *only if* it is *not* assigned this extension $|(3)|_3$. In other words, $u$ is guaranteed to remain the unique urelement that represents the class-union of the complement of $c$ and the complement of $d$.

Another instructive example is found in the two formulae '$x$ is a set' ($Sx$) and '$x$ has a member' ($\exists y(y \in x)$). Interpreted at stage 0, these predicates have the same extension, namely the pure sets $V_\kappa(C)\backslash(C)$. However, there will be two different urelemente $c$ and $d$ such that $\Xi(C) = \ulcorner Sx \urcorner$ and $\Xi(d) = \ulcorner \exists y(y \in x)\urcorner$ but $\mu(c) = \langle \alpha, 0\rangle$ and $\mu(c) = \langle \beta, 0\rangle$. In other words, there will be two different urelemente corresponding mapped to the same extension $V_\kappa(C)\backslash(C)$. Fortunately, though, due to the lexicographical, i.e. strict linear ordering of the formulae we can assume, without loss of generality, that $c \ll d$. Therefore, $d$ will not satisfy the condition imposed on membership in definition 4 ($|\Xi(d)| = |\Xi(c)|$ and $c \ll d$), hence $c$ witnesses the second conjunct).

This reasoning can be generalized to a proof that the construction does not *overgenerate*.

**Proposition 6.** The classes of the models $\mathfrak{M}_\alpha$ are extensional. For any $c, d \in C$ and any $\alpha$, if $c, d$ are ascribed members $(\exists x(x\, e_\alpha\, c \wedge x\, e_\alpha\, d))$ then

$$\forall x(x\, e_\alpha\, c \leftrightarrow x\, e_\alpha\, d) \rightarrow c = d$$

*Proof.* See appendix A.1. □

Moreover, the recursive definition of the $\mathfrak{M}_\alpha$ captures the intuitive idea of the construction as a step-by-step process in the course of which *more and more* classes are defined. The range of the membership relation increases strictly. More generally,

**Proposition 7.** for any membership sequence, the range of the membership relation increases monotonically, in the sense that for every $\alpha, \beta < \lambda$,

$$\text{If } \mathrm{rn}(e_\alpha) \subseteq \mathrm{rn}(e_\beta) \text{ then } \mathrm{rn}(e_{\alpha+1}) \subseteq \mathrm{rn}(e_{\beta+1})$$

*Proof.* See appendix A.2. □

The monotonicity of the membership sequences also ensures the existence of least fixed point membership relations $e_\lambda$. Every class definable at this stage is already represented in the range of $e_\lambda$ (if $x \in |\Xi(y)|_\lambda$ then $xe_\lambda y$). The model $\mathfrak{M}_\lambda$ corresponds to the final round of God and Gabriel's dialogue (1.1), at the end of which Gabriel has fully understood class membership. How large this terminal ordinal $\lambda$ really is depends on how quickly the urelemente $C$ are used up. This again is a matter of which predicates $\phi(x)$ the classes are mapped to, and therefore depends on $\Xi$.

Fine, however, prefers to fix the terminal ordinal directly. For this, he introduces the notion of *class-inaccessibility.* $\lambda$ is class-inaccessible if there is no ordinal $\alpha < \lambda$ such that for any membership sequence $\mathfrak{M}_\alpha$ defines a well-ordering of order-type $\lambda$ (if there is such an ordinal $\alpha$, $\lambda$ is *accessible*).[4]

Fine proposes to focus on the *least* such class-inaccessible ordinal [Fine, 2005, p. 556]. Fine motivates this choice from the heavenly dialogue by which he had already illustrated the construction of the $\mathfrak{M}_\alpha$. God leaves it to Gabriel to decide how long their question-and-answer game continues.

Clearly, though, Gabriel cannot overview the construction as a whole. Nonetheless, there is a way for him to fix the length of the dialogue from 'within'. The membership

[4]In fact, Fine's notion of class-accessibility is somewhat weaker since it quantifies only over *regular* models in the sense of §1.2.2 below.

relation $e_\alpha$ allows to formulate well-orderings on the universe. Each of these fixes an ordinal (their order type) and thus may be used by Gabriel to request a membership relation $e_\alpha$.

If the construction proceeds up to the least class-inaccessible ordinal, therefore, it is continued as long as Gabriel may possibly wish. Membership sequences of this length thus reflect God's '... well-known love of freedom' [p. 556].

This is a nice picture. A more sober reason to let $\lambda$ be the least class-inaccessible ordinal is found in the following remark.

> Just as set-inaccessibility represents a natural closure condition for the formation of sets, so class-inaccessibility represents a natural closure condition for the definition of classes. [p. 557]

Class-inaccessibility, Fine suggests, transposes the usual set-theoretic notion into the class-theory of the models $\mathfrak{M}_\alpha$, and thus allows for the following analogy. Just as the least inaccessible cardinal is a natural upper bound to the cumulative hierarchy, the model of the least *class*-inaccessible cardinal completes the development of class-membership.

In Fine, the properties of the terminal models also depends on the cardinality or 'height' of the universe $V_\kappa(C)$. Namely, since the range of the membership relation keeps increasing, the size of $V_\kappa(C)$ constitutes an upper bound to the terminal ordinal $\lambda$.[5] However, since above the cumulative hierarchy has been constructed by means of the restricted power-set operation $\mathcal{P}^{<\kappa}$ (p. 4), this complication may be neglected. In the present context, the size of the universe just is $\kappa$.

Fine's construction was motivated by the logical conception of class: Given some concept, there is a class of everything that falls under this concept. The challenge was not only to ensure that but also to explain why certain concepts, like the one of not being one's own member, do not give rise to classes.

The theory of the model $\mathfrak{M}_\lambda$ is Fine's proposal of a class theory. It does a remarkable job. Since it is the fixed point of a membership sequence, for any concept definable in $\mathfrak{M}_\lambda$ there is a corresponding class in the range of $e_\lambda$. In fact, the theory is closed under the Boolean operations, most interestingly union and complementation. Moreover, Fine's model-theory at its core implements the logical conception of class. The function $\Xi$, in terms of which a membership sequence is defined, maps concept to their extensions.

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[5] Clearly, the domain of $(e_\lambda)$ cannot exceed the universe.

Technically the challenge has been answered. We have a way of deciding which concepts fix a class — those definable in the model. But, the real philosophical work is yet to be done: Why does Fine's construction work? Eventually, I will propose that Fine's construction works because it reflects the groundedness of classes. First, however, let me look at how Fine justifies his construction. It will turn out that Fine's motivation makes implicit use of the idea of his class membership being grounded in set membership. Spelling out Fine's philosophical account will therefore naturally lead to my case that his theory is a theory of grounded classes.

## 2. Accounting for Fine's Class Theory

Above (§ 1.1), I motivated Fine's construction from the logical conception of class. According to it, classes are concept-extensions, containers that collect all and only the objects which fall under a given predicate. However, classes are first-order objects themselves, such that some device is needed to rule out the paradoxical instances.

In the preceding section I have presented Fine's theory of classes and its models. For their philosophical motivation I had distinguished between naive comprehension and naive membership — now I turn to clarify this idea.

### 2.1. Loosening the Vicious Circle Principle

Fine contrasts his proposal with predicativism [Fine, 2005, pp. 569nn]. The difference he emphasizes is not technical but lies in the philosophical account of (real) definition. Fine's theory and predicativist class theory are motivated by two different interpretations of the intuitive thought that definitions must not be circular.

Predicativism is motivated by the view that the definition of a class is both necessary and sufficient for its existence [p. 569]. In this sense, a definition *introduces* what it defines. If so, then the definition must not presuppose the object to be defined. Formally, the defining term must not involve quantification over a domain to which this object belongs. In other words, predicativists endorse the *Vicious Circle Principle* ('VCP').[6] A class comprehension schema that obeys the VCP, however, cannot satisfy the logical conception of class. Many concepts that are expected to define a class do not meet the

[6]To apply Gödel's useful distinction between three different understandings of the VCP [Gödel, 1990, p. 135], the principle at hand is the first of these, pertaining to the phrase 'definable only in terms of'.

VCP. One example is the concept of being a complement. The corresponding formula '$\exists y \forall z(z \in x \leftrightarrow z \notin y)$' involves two quantifiers. Neither the existential nor the universal quantifier, however, are in any way restricted - they range over the whole universe. Especially, their range includes the class to be defined. Thus, the concept does not meet the requirement imposed by the VCP. No predicativist theory of classes can satisfy the logical conception.

Fine, now, holds that to define a class $c$, its existence may be assumed [p. 570]. Definitions need not carry the burden of existence. Instead, the definition of $c$ serves to distinguish it from other objects, to single it out from the rest of the universe. In a slogan, definitions don't introduce, they *identify*. On this view, Fine argues, a class definition does not presuppose '... the objects in the range of its variables but the extension of its membership predicate' [ibid.].

I think it's worthwhile spelling out this inference. On one hand, if a definition is not necessary for the existence of $c$, then quantification over the domain of $c$ does not make the definition circular. Thus, Fine's alternative account of definition need not satisfy the VCP; it allows for impredicative classes.

On the other hand, though, the definition of $c$ must provide a criterion for any object to be $c$, respectively to differ from it. Needed, in other words, is a first level *identity criterion*:

$$\forall x(x = c \leftrightarrow \phi(x))$$

The definition of $c$ presupposes one instance of this schema.

If the object to be defined is a class in the logical sense, that is the extension of some concept $\psi(x)$, then an identity criterion is provided, if any sentence $\psi(t/x)$, for $t$ some object of the domain, has a determinate truth value. This, again, can only be the case if every atomic sentence $x \in y$ in $\psi(t)$ is either true or false. In sum, on Fine's identification view of definitions, the definition of a class $\{x|\psi(x)\}$ presupposes that the membership relation is interpreted for all its occurrences in $\psi$. This is how it presupposes '... the extension of its membership predicate' [ibid.]

Since on the present account of definition the universe of classes as a whole can be presumed, the relation's field as such is unproblematic. What needs be determined carefully is just how the classes are distributed over its domain and range — in other words, the extension of '$\in$'. And it is at this point that the difference between a *naive* and a *sophisticated* understanding of membership comes into play. The naive class theorist assumes that for every class it can be determined whether any object is among its members. Alas, this ‘anything goes’ approach leads straight to paradox. Or, back to a diminished predicativist universe.

If neither is wanted, therefore, the notion of class-membership needs development. What, however, distinguishes naive from elaborate, vicious from safe membership? Fine provides an ingenious formal definition, but he does not spell out its philosophical foundation. Thus, his proposal may seem *ad hoc*, and the reliability of his method uncertain.

Let me attempt to explain why the membership relation of Fine’s terminal models is of the right kind.

## 2.2. Fine’s Approach is a Groundedness Approach

If we want to leave behind naive membership, we must no longer presume the range of $\in$ to include every class. Instead, we need to build up the range carefully, and ensure that whenever a class enters the range, it is provided with an identity criterion. How do Fine’s membership sequences succeed in this?

The difference between the membership relation of the paradoxes and that of Fine’s impredicative[7] but consistent class theory is that Fine’s terminal membership relation $e_\lambda$ is developed from ordinary set-theoretic elementhood in a step-by-step manner. These two aspects of Fine’s construction, its departure from ordinary ZFC and the stepwise proceeding, together provide his classes with definite identity. Thus, on Fine’s account, they ensure safe definition and consistent class theory. Let me explain why.

The base case is, appropriately, basic. Fine’s initial model $M_0 = V_\kappa$, as any transitive set, validates the ZFC axiom of Extensionality, which is just a simple criterion of identity for sets.

Second, Fine’s membership sequence as defined above (p. 10) preserves the definite identity of classes; it ensures that whenever a class is defined, this class is already endowed with an identity criterion. The limit case is safe if the successor stages are, since here nothing is added to the range of $\in$; we just collect the previous stages. At successor stages, the class $c$ enters the range of the membership relation only if it represents the extension of a concept definable in the previous model. For this predicate $\phi(x)$, therefore, it is determined, for any $t$, whether $M_\alpha \models \phi(t)$ or not. Thus, the fomula $\forall x(x = c \leftrightarrow \phi(x))$ will serve as an identity criterion for $c$. Since only such classes

[7] In the strict sense that it does not obey the VCP.

$\{x|\phi(x)\}$, for concepts $\phi(x)$ definable in $M_\alpha$, are added to the range of membership in $M_{\alpha+1}$, every class of Fine's theory is ensured to come with an identity criterion.

I now propose to sum up the foregoing as follows. Fine's membership relation $e_\lambda$ is *grounded* in ordinary set-theoretic elementhood. It is its groundedness that distinguishes it from the naive membership relation and thus allows Fine to endorse impredicative instances of comprehension.

# 3. Groundedness

Notions of groundedness have figured in the literature on the semantic paradoxes.[8] However, I have in mind a more general conception. It does not only apply to sentences, propositions, or to the truth-values of sentences or propositions. For any domain $D$ and any collection of $D$-objects we may ask whether these objects are grounded; more precisely, whether they are grounded in some designated collection $G$.

What does it mean for $S$ to be grounded in $G$? I take groundedness to be a philosophical notion. Therefore, let me first give an informal picture. On its basis I will then develop a mathematical model of groundedness that allows me to specify my thesis that Fine's class-membership relation is grounded in ordinary set-theoretic elementhood.

## 3.1. Groundedness, Philosophically

In a nutshell, $S$ is grounded in $G$ if you arrive at $S$ from $G$ by applying successively some operation $\gamma$ of the right, *grounding* kind. This operation and its iteration are two distinct aspects of the notion of groundedness. In order to spell it out I will consider these aspects separately, explaining first the grounding character of $\gamma$, and then say something more about its iteration.

You may think of $\gamma(G)$ as a construction from $G$, but only metaphorically. The proposed account of groundedness is meant to be thoroughly realist. Groundedness, as I think of it, is an objective property. Objects are grounded in virtue of how the world is like, independently of our constructive abilities.[9] Consequently, there are no limits as to how $\gamma(G)$ is computed but for one crucial constraint: the only input is $G$.

[8] [Herzberger, 1970, Kripke, 1975, Yablo, 1982, McCarthy, 1988, Maudlin, 2004, Leitgeb, 2005]

[9] For the sake of readability, I will nonetheless make frequent use of construction talk; this will always be merely metaphorical.

However, the interesting collections of objects grounded in $G$ are not obtained in a single step. Instead, the operation $\gamma$ is iterated. Once the objects $\gamma(G)$ are obtained, they may be used themselves as input for $\gamma$. Thus, another collection $\gamma(\gamma(G))$ is generated, which again is grounded in $G$; and so on. Notice that since the grounding operation is iterated, some conception of ordinal number is built into the notion of groundedness. This is the 'step-by-step' aspect of groundedness that I have found in Fine's construction.

I do not think that while $\gamma$ is iterated, the collections necessarily become bigger and bigger, or richer and richer. $\gamma(X)$ may well be just a fragment of $X$. Nonetheless, every new collection obtained from applying $\gamma$ is *grounded* in the starting point $G$. Especially, $S$ being grounded in $G$ does not mean that no new collection $\gamma(S) \neq S$ could be obtained from it (groundedness does not imply being a fixed point).

Since groundedness does not depend on any subject to carry out all the iterations, there are no constraints as to how many times the operation is applied. Therefore, the iteration of $\gamma$ is continued beyond limit stages and the notion of ordinal numbers at work in groundedness is fully transfinite.

In sum, a collection $S$ is grounded in $G$ if there is an operation $\gamma$ such that, if you start from $G$ and iterate $\gamma$ transfinitely often, whereby at each stage you only use what the previous stage has given, you arrive at $S$.

## 3.2. A Formal General Theory of Groundedness

I now turn to formalize the intuitive idea of groundedness sketched in the previous section. This formalization is meant to model (in the scientist's sense) groundedness. I do not intend it to carve out the notion's essence. The definition suggested below provides handy tools to analyze various supposed cases of groundedness. Nonetheless, it is the philosophical idea of groundedness sketched above that I take to be basic.

For $D$ some domain, $S \subseteq D$ is grounded in $G \subseteq D$ if there is an operation $\gamma$ on $D$ such that $S$ is obtained from $G$ by iterated application of $\gamma$. This iterative aspect of groundedness is spelt out best as follows. $G$ and $\gamma$ define a function $F : \Omega \mapsto \mathcal{P}(D)$:

$$
\begin{aligned}
F(0) &= G \\
F(\alpha+1) &= \gamma(F(\alpha)) \\
F(\beta) &= \bigcup_{\alpha<\beta} F(\alpha), \text{ for limit } \beta
\end{aligned}
$$

By requiring $S$ to be $F(\alpha)$ for some ordinal $\alpha$ we thus give formal expression to the intuitive thought that $S$ arises from $G$ by iterated application of the operation $\gamma$.

It is somewhat more intricate to spell out what is required of the operation $\gamma$. The informal idea is that determining $\gamma(X)$ should not require us to, speaking figuratively, go beyond $X$. The task is to model this intuitive picture in mathematical terms.

Abstracting from intuition, $\gamma$ is a mapping on the domain $D$, a way of arriving from some $S$ at another collection $S'$. What enables $\gamma$ to ground $S'$ in $S$ is how we arrive from $S$ at $\gamma(S)$. For one, $\gamma(S)$ should be obtained in an orderly manner, following a general rule that treats every $S$ the same. Thus, the computation metaphor evoked above carries further than it may have initially seemed. Not that the grounding operation $\gamma$ really was a computation, but operations of the right, grounding kind are well described in computational terms. Therefore, I choose a recursion-theoretic setting to develop a formal definition of groundedness.

Moreover, thinking of $\gamma$ as an algorithm suggests a neat way of formalizing the intuitive idea that the grounding operation 'only uses what it is given'. This informal constraint on $\gamma$ is well rephrased as saying that the only input of this computation is $X$. Recursion-theory, now, provides the means to make explicit what is used during a computation. If $f(x)$ can be computed using only another function $g$ as 'oracle' then $f$ is said to be recursive in $g$. The *grounding* character of $\gamma$ may therefore be formally modelled as the recursiveness of $\gamma(X)$ in $X$.

At a closer look, however, this is too restrictive. My realist understanding of groundedness does not require $\gamma(X)$ to be computed by a Turing machine from $X$. There is no need to assume that the construction of $\gamma(X)$ should be completed after finitely many steps. As explained above, the conception of groundedness I wish to capture comes with all the ordinals. Accordingly, I prefer not to put any restrictions on how long it may take to apply $\gamma$ to $X$. Fortunately, generalized computability theory allows me to put this informal idea into mathematical terms.

Take the usual Kleene equations (initial functions, composition, minimal recursion) but allow for derivations of length $\beta$, for any limit ordinal $\beta$. A function is $\beta$-recursive if it can be deduced from the usual equations in $\beta$ many steps. Thus, the generalized sense of recursiveness relevant for my formal definition of groundedness is what is known in the literature as *$\beta$-recursion* [Shore, 1978, Sacks, 1990, Chong and Friedman, 1999].

However, I confine myself to a notion of reducibility simpler than what has become standard in the recursion theoretic literature.[10] Following [Shore, 1975, Shore, 1978] I will speak of one set A being '$\beta$-calculable' from another set $B$ if $c_A$ can be deduced from the recursive equations and $c_B$, however many steps this may take.

In sum, I propose the following formal definition of groundedness.

**Definition 8.** $S \subseteq D$ is *grounded* in $G \subseteq D$ iff

1. there is an operation $\gamma : D \mapsto D$ and a limit ordinal $\beta$ such that for any $X \subseteq D$, $\gamma(X)$ is $\beta$-calculable from $X$ and
2. $S = F(\beta)$ for some ordinal $\beta$ where $F(0) = G$, $F(\alpha + 1) = \gamma(F(\alpha))$ and $F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha)$

Otherwise, it is *ungrounded*

Below, I will show that Fine's terminal membership $e_\lambda$ is grounded in set-theoretic elementhood $e_0$ in just this sense. Before returning to Fine's class theory, however, it will be useful to consider some paradigm cases of groundedness, and show how the definition applies to them.

### 3.2.1. Well-Founded Sets

To warm up, let me show how my formal account of groundedness applies to the cumulative hierarchy of the well-founded, pure sets (that is, $\mathbf{V}$). Intuitively, the $V_\alpha$'s are grounded in the empty set. Therefore, let $D$ be $V_\kappa$ for some inaccessible $\kappa$ and $G = \varnothing$. $\gamma$ becomes the power-set operation $\mathcal{P}(S)$, which can be spelt out as follows:

$$y \in \mathcal{P}(S) \text{ iff } \forall u(u \in y \rightarrow u \in S)$$

The universal quantifier is bound by $y$, and '$u \in S$' is trivially recursive in $S$. Therefore, $\mathcal{P}(S)$ is recursive in $S$ and grounds every $V_\alpha$ in $\varnothing$, in the specified sense of definition 8.

### 3.2.2. Kripke Truth Theory

Kripke (1975) describes a family of truth theories. I will focus on the theory of his least fixed point model in strong Kleene logic. This theory is widely taken to be a paradigm case of groundedness. When motivating his construction, Kripke draws heavily

[10] I may do so because for the present purpose it does not matter that $\beta$-calculable functions may need more than $\beta$-many steps to be computed.

on groundedness intuitions. Therefore, the adequacy of my proposed formal definition of groundedness clearly depends on whether it applies to Kripke's theory.

Let $D_k$ be the set of (the codes of) sentences of the language of arithmetic plus truth predicate, and $G_k$ collect the (codes of) truths of the standard model $\mathfrak{N}$ ($G_k = \{\ulcorner\phi\urcorner | \mathfrak{N} \models \phi\}$). Consider extensions of $\mathfrak{N}$ by a partial interpretation $\langle E, A\rangle$ of the truth predicate ($\mathfrak{N}(E, A)$). Define a function $\gamma_k : \mathcal{P}(D_k) \mapsto \mathcal{P}(D_k)$ such that

$$\gamma_k(S) = \{\ulcorner\phi\urcorner | \mathfrak{N}(S, \{\ulcorner\psi\urcorner | \ulcorner\neg\psi\urcorner \in S\}) \models \phi\}$$

$\gamma_k$ is a monotone operation on $D_k$, and Kripke's truth theory can be described as $F_k(\omega_1^{CK})$. Thus, to show that the theory of this least fixed point is grounded in $G$, in the formal sense of definition 8, I only need to explain how $\gamma_k(S)$ is calculable from $S$.

In fact, $\gamma_k(S)$ is recursive in $S$. Notice that the definition of $\gamma_k(S)$ can be spelt out in terms of the (primitive) recursive encoding functions of pairs and syntax, the only non-recursive elements being a universal quantification over $S$ and the model-theoretic truth predicate $\mathfrak{N}(S) \models x$. Both, however, can be computed once $S$ is available.

Thus, the formal definition of groundedness has proven applicable to prominent cases of groundedness. Equally important, however, it is to show that the definition proposed does not apply to such cases that resemble the paradigm cases above closely but do not themselves reveal groundedness. The second example of intuitive groundedness I have given above was Kripke's theory of the minimal fixed point. Now, I turn to an intuitively *ungrounded* variant of it and show that it is also ungrounded in the sense of definition 8.

### 3.2.3. An Ungrounded Fixed Point Truth Theory

Let $D_k$ and $G_k$ be as above. $D_k$ is the set of sentences of a first order language of arithmetic plus truth predicate, and $G_k$ the set of first order arithmetical truths. In this familiar setting define a deviant operator $\gamma_d$. Take the ordinary Kripke fixed point theory considered before. Its sentences can be enumerated, such that $\phi_n$ is the $n$th according to this lexicographical ordering. Given this, define

$$\gamma_d(x) = x \cup \{\phi_n | n \leqslant \text{ the number of sentences in } x \text{ that contain the truth predicate}\}$$

such that $\gamma_d(G_k)$ is $G_k \cup \{\phi_0\}$, $\gamma_d(G_k \cup \{\phi_0\}) = G_k \cup \{\phi_0, \phi_1\}$ and so on.

$\gamma_d$, too, is monotone and we may consider its least fixed point. Clearly, the consistency of Kripke's theory is inherited to it. Nonetheless, its consistency is *not* grounded in the consistency of the non-semantic base theory. The operation $\gamma_d$ is of the wrong kind, it does not ground its range in $G_k$.

Recall that according to my informal sketch in section 3.1 the grounding operation may only use what it is given as input and must not rely on other resources. The fake Kripke jump $\gamma_d$, however, at any stage makes use not only of what has been obtained before but also goes through the original Kripke fixed point to find the sentences next in the lexicographical order.

The intuitive difference between the grounding Kripke jump and $\gamma_d$ can also be seen as follows. Motivating his construction Kripke describes how a speaker of the non-semantic base language learns to use the truth predicate if she is taught the operation $\gamma_k$ (and given some time). It seems plausible that a speaker, at any rate an idealized speaker as in Kripke's thought experiment, could learn to use $\gamma_k$. All there is to $\gamma_k$ is applying the Strong Kleene evaluation scheme to sentences containing truth.

$\gamma_d$, however, is much less suitable as the key to learn truth talk. For one, its definition seems too complicated to lend plausibility to the picture of someone applying the operation to her arithmetical knowledge. Moreover, even if an idealized subject had mastered $\gamma_d$ then there would be no more need for her to go through the construction of its fixed point, not even one step. Having grasped the definition of $\gamma_d$ she already commands the full Kripke truth theory; putting it into a lexicographical order is no further obstacle.

My formal definition 8 captures well this intuitive difference between the Kripke jump and $\gamma_d$. As seen above, $\gamma_k$ meets the requirement that the definition imposes on the grounding operation — $\gamma_k(X)$ is recursive in $X$. Not so $\gamma_d(X)$. Recall that $\gamma_k(X)$ is the extension of $X$ by the sentence $\phi_n$ that stands at the $n$th position in the lexicographical ordering of the original Kripke fixed point theory, where $n$ is the number of sentences with truth predicate that occur in $X$. However, just which sentence this is cannot be determined from $X$. $X$ by itself only tells us, speaking figuratively, which row to look up in the list, but it does not give a hint what is found there. To know this, we first need to know the sentences of Kripke's theory, all at once. Only by consulting the Kripke fixed point $F_k(\omega_1^{CK})$, therefore, we can find out which sentence is to be added next. In other words, the computation of $\gamma_k(X)$ from $X$ requires the oracle $F_k(\omega_1^{CK})$.

Thus, $\gamma_d$ does not meet the requirements of my definition 8. The theory of its least fixed point, although consistent, is ungrounded in the formal sense of the definition. This definition, therefore, has not only proved correct with respect to cases of groundedness but as well with respect to cases of ungroundedness.

I will therefore assume the adequacy of definition 8 and on its basis now turn to show that Fine's class membership relation $e_\lambda$ is grounded in set-theoretic elementhood $e_0$.

### 3.3. Fine's Terminal Membership is Grounded

Recall the model-theoretic construction of section 1.2 and its modification on page 10. Let $D$ be the set of pairs $\langle x, y\rangle$ for any $x, y \in V_\kappa(C)$ ($D = V_\kappa(C)^2$). Further, let the ground $G$ be the set-theoretic membership relation $e_0 = \{\langle x, y\rangle | y \in V_\kappa(\varnothing) \text{ and } x \in y\}$. Finally, define the operation $\gamma_F : D \mapsto D$ as follows.

$$\gamma_F(S) = \begin{cases} e_{\alpha+1} & \text{if } S = e_\alpha \text{ for some } \alpha \\ S & \text{otherwise} \end{cases}$$

$\gamma_F$ and $G$ define a function $F$ from the ordinals into the power-set of $D$ such that Fine's designated class-membership relation $e_\lambda$ is $F(\lambda)$, for the least class-inaccessible ordinal $\lambda$ (see p. 12 above). To show that $e_\lambda$ is indeed grounded in $G$ in the sense of my definition 8 I only need to show that for any $S$, $\gamma_F(S)$ is calculable from $S$.

If $S$ is not the membership relation of some model in Fine's membership sequence then $\gamma_F(S) = S$ and therefore trivially recursive in $S$. If $S$ is some $e_\alpha$ then (compare definition 4)

$$\gamma_F(e_\alpha) = \{\langle x, y\rangle | x\, e_\alpha\, y, \text{ or } x \in |\Xi(y)|_\alpha$$
$$\text{and for any } \beta \leqslant \alpha \text{ there is no } c \in C \text{ such that } c \ll y \text{ and } |\Xi(y)|_\alpha = |\Xi(c)|_\beta.$$

The relation '$x \in |\Xi(y)|_\alpha$' is recursive in $e_\alpha$, since it abbreviates $\mathfrak{M}_\alpha \vDash \phi(x)$, for $\phi = \Xi(y)$, and truth in the structure $\langle V_\kappa(C), e_\alpha\rangle$ is calculable from $e_\alpha$.[11]

My discussion of Fine's proposal above showed that the definition of $e_{\alpha+1}$ needs the following constraint:

$$\forall\beta \leqslant \alpha \neg\exists c \in C(c \ll y \wedge |\Xi(y)|_\beta = |\Xi(c)|_\beta) \quad (4)$$

[11] Logical truth and identity is recursive and the membership relation of $\mathfrak{M}_\alpha$ is trivially recursive in $e_\alpha$. Set-hood, finally, can be defined in terms of $e_\alpha$, quantifying over ordinals less than $\alpha$: $Sx$ iff $\exists u(ue_\alpha x$ and $\forall\beta < \alpha(ue_\beta x))$.

On the intended interpretation, this formula expresses a function of $y$ that is $\lambda$-calculable from $e_\alpha$. Since there are only $\kappa$ many urelemente, and $\lambda > \kappa$, the set of urelemente $C$ is $\lambda$-recursive.[12] Therefore, not only is the universal quantifier over urelemente implicit in (4) $\lambda$-recursively bounded; '$\ll$', too, is defined in terms of basic operations on the ordinals and the function $\mu : C \mapsto \kappa \times \lambda$.

Since the recursive encoding of pairs of natural numbers in the natural numbers is straightforwardly generalized to the ordinals, $\mu$ is $\lambda$-recursive. Therefore, $\gamma_F(X)$ is calculable from $X$. This ensures the operation $\gamma_F$ to *ground* its output. Especially, Fine's class membership $e_\lambda$ is grounded in the set-theoretic elementhood relation.

## 4. Comparing Groundedness with Fine's Regularity

Having shown that Fine's class membership relation is grounded in ordinary set membership, I now turn to compare my notion of groundedness with his concept of *regularity* (see p. 5 above). Fine draws an analogy between his regular models and the well-founded models of ZF [Fine, 2005, p. 554]. In section 3.2.1 above, I have found that the well-founded sets are grounded. Therefore, Fine's analogy suggests that by focusing on regular mappings $\Xi$ he aims for grounded class membership. However, my notion of groundedness exceeds regularity. What my argument from the previous section has shown is that *any* model of a membership sequence is grounded in the set-theoretic elementhood relation. So, a grounded membership relation may not be regular in Fine's sense. To use Fine's own example [Fine, 2005, p. 553], consider a $\Xi_i$ that maps $c \in C$ to '$x \neq d$' and $d \in C$ to '$x \neq c$'. By the extensionality of Fine's classes (proposition 6 above), '$x \neq d$' is equivalent to '$\forall y(y \in x \leftrightarrow y \in d)$'. Therefore, at any stage $\alpha$ where $d$ is not in the range of the membership relation, $|\Xi_i(c)|_\alpha$ is empty, such that $c$ is not added to the range of $e_\alpha$. In other words, $c$ will be ascribed members only when $x \in d$ is satisfied by some objects of $V_\kappa(C)$ However, for $d$ to enter $\mathrm{rn}(e_\beta)$ for $\beta \leqslant \alpha$ $c$ would have entered the range of the membership relation first —which contradicts the assumption that $\alpha < \beta$. The same reasoning applies to $\Xi(d)$; hence, neither $c$ nor $d$ ever enter the range.

More generally, irregular $\Xi$ imply that some urelemente never are ascribed members in the sense of some membership relation $e_\alpha$. But this does not mean that the corresponding membership sequences do not complete. They are still monotone (proposition 7), and

[12] Simply generalize the standard reasoning that ensures finite sets to be recursive.

therefore still have a least fixed point. Especially, grounded but irregular membership relations still give rise to consistent class theories. Prominently, they do rule out Russell style antinomies. For, assume that there was such an $r = \{x|\neg x e_\lambda x\}$ that itself was in the range of $e_\lambda$. By the definition of $e_{\alpha+1}$, this can only be if for some $\alpha < \lambda$, $\delta_\alpha(r)$ is the extension $|x \notin x|_\alpha$ This subset of the universe, however, need be just $\{x|\neg x e_\lambda x\}$, which is $|x \notin x|_\lambda$, contrary to $\alpha < \lambda$. Hence, $r$ cannot be in the range of $e_\lambda$; '$\neg r e_\lambda r$' is a simple truth that does not give rise to any contradiction.

I conclude that Fine's definition of regularity does not capture groundedness ideas. Instead, the regularity of mappings $\Xi$ ensures the class membership relation to eventually include every urelement. For regular $\Xi$, namely, $\Xi(c)$ '... "bottoms out" in definitions that make no appeal to proper classes' [Fine, 2005, p. 554]. Consequently, for any $c$ there will always be a stage $\alpha$ where $|\Xi(c)|_\alpha$ is non-empty. By definition 4, $c$ will therefore enter the range of $e_{\alpha+1}$. Introducing a precise concept of groundedness and applying it to Fine's work thus has also allowed me to specify the role of the notion of regularity that he himself deploys in his paper.

## 5. Conclusion

Although some technical problems needed attention along the way, the present study was guided by a philosophical question. What is the reason that certain concepts which violate the Vicious Circle Principle may nonetheless be mapped to first order objects, that is, define classes? The answer I developed in the present essay is: classes are *grounded* in the cumulative hierarchy of sets. An impredicative concept defines a class, therefore, if it is grounded in set theory itself.

Fine's model-theoretic construction, as reformulated in my section 1.2, shows that impredicative class-comprehension is available. Fine also provides his work with philosophical motivation (§2.1). Although I did not find his account to be the principled explanation that the present study aimed for, it gave a starting point that I developed into a conception of groundedness (§§2.2, 3.1).

In order to clarify the informal account of section 3.1 I proposed a formal definition of groundedness in recursion-theoretic terms (§3.2). I argued for the adequacy of this definition on the basis of two intuitive cases of groundedness §§3.2.1, 3.2.2, as well as two non-examples §§**??**, 3.2.3. I showed that Fine's terminal membership is grounded in the precise sense of this definition. Finally (§4), I briefly compared my notion of groundedness with Fine's definition of *regular* membership sequences.

Fine's construction succeeds because it ensures that the membership relation in terms of which classes are defined is grounded in ordinary set-theoretic elementhood. Fine's theory of classes exemplifies how fruitfully the concept of groundedness can be applied; this I hope to have shown by the present study.

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# A. Appendix

## A.1. Extensionality of Proper Classes

Recall proposition 6 that ensured the extensionality of Fine's classes.

**Proposition.** For any $c, d \in C$ and any $\alpha$, if $\exists x(x\, e_\alpha\, c)$ then

$$\forall x(x\, e_\alpha\, c \leftrightarrow x\, e_\alpha\, d) \rightarrow c = d$$

*Proof.* Argue by induction on $\alpha$. If $\alpha = 0$ then the claim is vacuously true since no urelement is in the range of $e_0$. For $\alpha$ limit ordinal, $\mathrm{rn}(e_\alpha) = \bigcup_{\gamma<\alpha} \mathrm{rn}(e_\gamma)$ such that $(x\, e_\alpha\, c \leftrightarrow x\, e_\alpha\, d)$ only if $(x\, e_\gamma\, c \leftrightarrow x\, e_\gamma\, d)$ for some $\gamma < \alpha$, but then $c = d$ by the induction assumption.

Assume that $\alpha = \beta + 1$, $\exists x(x\, e_\alpha\, c)$ and $\forall x(x\, e_{\beta+1}\, c \leftrightarrow x\, e_{\beta+1}\, d)$. It cannot be that $x\, e_\beta\, c$ but not $x\, e_\beta\, d$. Namely, for $x$ not to be in the range of $e_\beta$ there would have to be a $\gamma < \beta$ such that $|\Delta'(c)|_\gamma = |\Delta'(d)|_\beta$ which contradicts the assumption that $d \in \mathrm{rn}(e_\alpha)$ $(x\, e_\beta\, c \rightarrow x\, e_{\beta+1}\, c \leftrightarrow x\, e_{\beta+1}\, d)$. Hence, there are two cases. Either (i), $x\, e_\beta c \leftrightarrow x\, e_\beta d$ and this implies, together with the induction assumption, $c = d$. Or (ii), $x \in |\Delta'(c)|_\beta \leftrightarrow x \in |\Delta'(d)|_\beta$ such that $|\Delta'(c)|_\beta = |\Delta'(d)|_\beta$. Assume that $c \neq d$ and without loss of generality $c \ll d$ – by the strengthened definition of $e_{\alpha+1}$ now $x$ cannot be $e_{\beta+1}\, d$, contrary to the assumptions $x\, e_{\beta+1}\, c$ and $x\, e_{\beta+1}\, c \leftrightarrow x\, e_{\beta+1}\, d$.

□

## A.2. The Monotonicity of the Membership Sequence

Recall proposition 7

**Proposition.** For every $\alpha, \beta < \lambda$,

$$\text{If } \mathrm{rn}(e_\alpha) \subseteq \mathrm{rn}(e_\beta) \text{ then } \mathrm{rn}(e_{\alpha+1}) \subseteq \mathrm{rn}(e_{\beta+1})$$

*Proof.* The claim follows easily from modest observations. For one, the case of $\alpha \geqslant \beta$ is trivial (observe that the range of $e$ increases). If $\alpha < \beta$, the following reasoning by double induction suggests itself. First, notice that for any $\alpha$ and limit $\beta$, $\mathrm{rn}(e_\beta) = \bigcup_{\alpha<\gamma<\beta}\mathrm{rn}(e_\gamma) \supseteq \mathrm{rn} e_{\alpha+1}$. For $\alpha = 0$, assume that $x \in e_1$. If $\beta = 1$ then clearly, $x \in e_{\beta+1}$. If $\beta = \gamma + 1$ and $x \in \mathrm{rn}(e_\gamma)$ then $x \in \mathrm{rn} e_{\beta+1}$, too. For successor $\alpha$ and $\beta = \alpha + 1$, $\mathrm{rn}(e_{\alpha+1}) \subseteq \mathrm{rn}(e_\beta) \subseteq \mathrm{rn}(e_{\beta+1})$ follows straightforwardly. The case for $\beta = \gamma + 1$ is established just like before. □